OSC

IB Mathematics

HL Option: Series & Differential Equations
International Baccalaureate
Revision Guide

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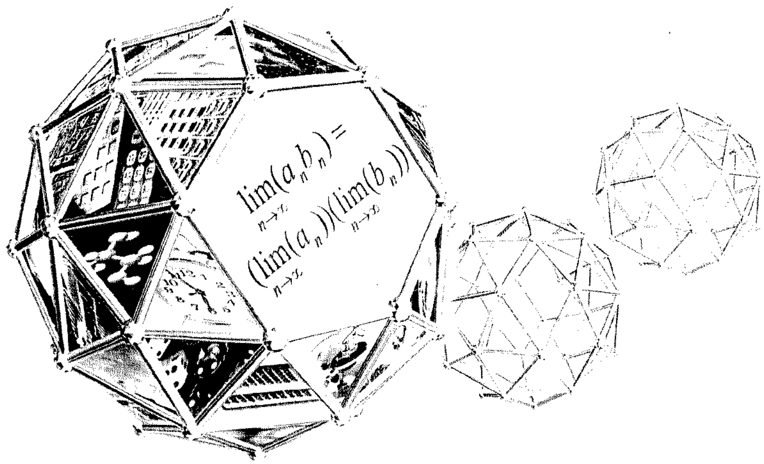


Oxford Study Courses

IB MATH HIGHER OPTION

SERIES AND DIFFERENTIAL EQUATIONS

This Option is a separate section of the exam and generally it is the last part of the syllabus to be covered, and therefore it is often the part least revised. The Options contain more advanced Mathematics than the core.

As It is worth 20% of the final marks, it merits special attention. The exam on the Option is a separate one-hour paper so you can really concentrate on it. There is no doubt that the first four sections of this part of the syllabus are difficult, but it is likely that many of the marks will be on the differential equations section and the Maclaurin and Taylor series, which are far more straightforward.

Do not forget to state the obvious in proofs, the examiners have marks to award for each step even if it seems unnecessary to you.

It is also worthwhile making sure that you can do all types of integration and to practice recognizing which ones are applicable to each question – an important skill in doing differential equations.

This is a revision guide and not a complete course, use your class notes and text books as well.

Make sure that you learn the definitions of all the terms, many words sound like ordinary English but have a very precise meaning in Mathematics.

You should follow through the worked examples if you are unfamiliar with the work and try and solve the questions without looking at the answers.

Please get in touch with me if you have any ideas for improvements that could be made to this booklet. I can be reached at pwendystevens@yahoo.com

10.1.1 Infinite sequences of real numbers

Your calculator is a very good tool to find whether the sequence is converging. Keep plugging in the numbers until you can see the limit to which the numbers are tending.
Many of the theorems are so straightforward that you seem to be stating the obvious, but don't worry: that is what the examiner is looking for.

> You have already come across finite and infinite sequences of numbers.
>
> The word **series** is used when individual terms are being summed. The word **sequence** is used when considering individual terms.
>
> We will use u_n for the nth term
> S_n for the sum of the first n terms

Arithmetic sequences have a constant difference d and first term u_1:
$u_1, u_1+d, u_1+2d, u_1+3d, \ldots\ldots\ldots\ldots$

Geometric sequences have a constant ratio r.
$u_1,\ u_1r,\ u_1r^2\ldots\ldots\ldots\ldots\ldots\ldots$
If the sequence has $-1 < r < 1$,
then the series will have a sum to infinity:

$S_\infty = \dfrac{u_1}{1-r}$ where r is the ratio.

e.g. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots\ldots = 2$

Recursive sequences
e.g. $u_{n+1} = u_n - 2u_{n-1}$ where $u_1 = 2$ and $u_2 = -3$.

This means the next term (u_{n+1}) is the present term (u_n) minus double the previous term (u_{n-1}).
$u_3 = u_2 - 2u_1$
$= -3 - 4 = -7$
$u_4 = u_3 - 2u_2$
$= -7 + 6 = -1$

A geometric sequence can be written as a recursive sequence like this:
$u_{n+1} = ru_n$

An arithmetic sequence can be written recursively like this:
$u_{n+1} = u_n + d$

Questions

a) Find S_∞ for $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} \ldots\ldots\ldots\ldots\ldots\ldots$
b) Find u_5 if $u_{n+1} = 3u_n + u_{n-1}$ and $u_1 = 2$ and $u_2 = 1$
c) Find u_4 if $u_5 = 8$ and $u_{n+1} = 2u_n + 3$
d) What type of sequence do these sequences show?
  i) $u_{n+1} = (1/3)u_n$
  ii) 1, 3, 3, 1
e) Can you write Fibonacci's sequence {1, 1, 2, 3, 5, 8,......} as a recursive sequence?

Answers
a) 0.75
b) 53
c) 2.5
d) i.) geometric sequence with common ratio $^1/_3$
  ii.) binomial sequence ^3C_r $r = 0,1,2,3$
e) $u_{n+1} = u_n + u_{n-1}$ $u_1 = u_2 = 1$

Definitions

The **limit** of a sequence is what the numbers are tending to:

Formal definition of the limit of a sequence:

A *sequence* $\{u_n\}$ converges to the limit L, if for any $\varepsilon > 0$, there is a positive integer N such that $|u_n - L| < \varepsilon$, for all $n>N$

This is essentially saying that however small an ε is chosen, eventually all the numbers in the sequence from a certain point onwards will be closer than ε to L.

> *Symbols and definitions that you need to know:*
> $\exists$ means "there exists"
> $\forall$ means "for all"
> $\Leftrightarrow$ if and only if
> $\{a_n\}$ a sequence with nth term a_n – the entire sequence is written as $\{a_n\}$
> A sequence is *bounded above* $\Leftrightarrow \exists M$ such that $a_n \le M\ \forall n$
> A sequence is *bounded below* $\Leftrightarrow \exists L$ such that $a_n \ge L\ \forall n$
> A sequence is *bounded* $\Leftrightarrow \exists K$ such that $|a_n| \le K\ \forall n \Leftrightarrow \{a_n\}$ is bounded above and below
> $\{a_n\}$ is positive $\Leftrightarrow a_n > 0\ \forall n$
> $\{a_n\}$ is non-negative $\Leftrightarrow a_n \ge 0\ \forall n$
> $\{a_n\}$ is negative $\Leftrightarrow a_n < 0\ \forall n$
> $\{a_n\}$ is non-positive $\Leftrightarrow a_n \le 0\ \forall n$
> $\{a_n\}$ is increasing $\Leftrightarrow a_{n+1} \ge a_n\ \forall n$
> A sequence is strictly increasing when each term is larger and not equal to the previous one.
> $\{a_n\}$ is decreasing $\Leftrightarrow a_{n+1} \le a_n\ \forall n$
> A sequence is strictly decreasing when each term is smaller and not equal to the previous one.
> A sequence is monotonic if it is either increasing or decreasing
> $\{a_n\}$ is alternating $\Leftrightarrow$ the terms alternate in signs.

N.B. An increasing sequence which is bounded above (i.e. bounded monotonic), must tend to a limit. Similarly, a decreasing sequence. which is bounded below must tend to a limit.

10.1.2 Limit theorems

You need to learn all of these precisely.

Limit of sum theorem

If $\{a_n\}$ and $\{b_n\}$ are real sequences and the limits exist, then $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

Difference theorem

If $\{a_n\}$ and $\{b_n\}$ are real sequences and the limits exist, then $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$

Product theorem

If $\{a_n\}$ and $\{b_n\}$ are real sequences and the limits exist, then $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$

Quotient theorem

If $\{a_n\}$ and $\{b_n\}$ are real sequences and the limits exist, then $\lim_{n\to\infty}(a_n \div b_n) = (\lim_{n\to\infty} a_n)\div(\lim_{n\to\infty} b_n)$ providing $\lim_{n\to\infty} b_n \neq 0$

Squeeze theorem

If $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences where all the terms in the three sequences, with $a_n \le b_n \le c_n$ $\forall n$ and $a_n \to L$ and $c_n \to L$, then $b_n \to L$.

Worked example question

Find the limit:

$$\lim_{n\to\infty}\frac{3n^3-2n^2+5n-3}{5n^3+3n^2-6n+2}$$

$$\lim_{n\to\infty}\frac{\frac{3n^3-2n^2+5n-3}{n^3}}{\frac{5n^3+3n^2-6n+2}{n^3}}$$

As n tends to infinity, the fractions with a denominator of n tend to 0 resulting in the final answer

$$\lim_{n\to\infty}\frac{3-\frac{2}{n}+\frac{5}{n^2}-\frac{7}{n^3}}{5+\frac{3}{n}-\frac{6}{n^2}+\frac{2}{n^3}}=\frac{3}{5}$$

using quotient, sum and difference theorems.

Example questions

1. Use a calculator to examine the limits of the following as x tends to infinity:
   a)$(\ln x)/2x$ b) xe^{-x} c) $e^x/(x^3 + 3)$
2. Use the squeeze theorem to find the $\lim_{n\to\infty}\frac{\sin n}{n}$
3. Use the laws of limits to find the following if they exist. Which laws have you used?
   a) $\lim_{n\to\infty}\frac{n^2+n}{3n^2+n}$ b) $\lim_{n\to\infty}\frac{e^n-e^{-n}}{e^n+e^{-n}}$
   c) $\lim_{n\to\infty}\frac{4x-3}{x^2-4}$ d) $\lim_{n\to\infty}\left(\frac{n^{10}}{(n+1)^{11}}\right)$

Answers

1. a) 0 b) 0 c) tends to infinity
2. $\frac{-1}{n}\le\frac{\sin n}{n}\le\frac{1}{n}$
   $\frac{-1}{n}$ and $\frac{1}{n}\to 0$
   so $\frac{\sin n}{n}\to 0$
3. a) ⅓ quotient and sum
   b) 1, quotient, sum and difference
   c) 0 quotient and difference
   d) $\left(\frac{n}{n+1}\right)^{10}\left(\frac{1}{n+1}\right)$ product
   $\frac{n}{n+1}\to 1 \Rightarrow \left(\frac{n}{n+1}\right)^{10}\to 1$ and $\frac{1}{n+1}\to 0$ so the product tends to 0

10.1.3 Improper integrals

Improper integrals with infinity as a limit

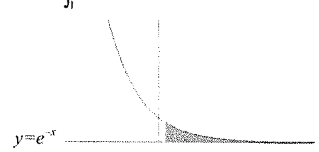
An integral is said to be improper if

a) $f(x)$ is not defined for at least one value in the range of integration (not in syllabus),

b) or the range of integration is infinite

Example

Find $\int_1^\infty e^{-x}dx$



The limit of e^{-x} as $x\to\infty$ is 0 and one can integrate this unbounded region to find the area:

$$\int_1^\infty e^{-x} = \lim_{X\to\infty}\int_1^X e^{-x}dx = \underset{X\to\infty}{Lim}[-e^{-x}]_1^X = \underset{X\to\infty}{Lim}(-e^{-X}+e^{-1}) = \frac{1}{e}$$

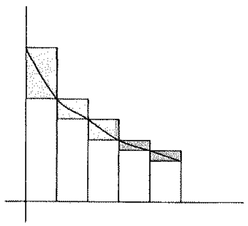
Example questions

Find the values a) $\int_4^{\infty} \frac{6}{x^3} dx$ b) $\int_4^{\infty} \frac{1}{16+x^2} dx$

Answer

a) 3/16 b) $\frac{1}{16}\pi$

10.1.4 The integral as a limit of a sum; lower sum and upper sum



It is clear from the diagram that the area under the curve is somewhere between the total area of the complete rectangles (which are called the "circumscribed" rectangles) and the total area of the lower rectangles (which are called the "inscribed" rectangles).

The narrower the rectangles are the closer the rectangles will be to the actual area.

In Newton's time the $\int$ sign was shorthand for "s" – the sum, and we still use this notation.

Each rectangle is of height *y* and width dx so the total area is $\int y\, dx$.

Question

a) Find the area of the inscribed rectangles under the curve $y = x^2 + 2$ from $a = 0$ to $a = 1$ with $n = 4$ rectangles *i.e.* the lower bound.
b) Find the area of the circumscribed rectangles over the curve $y = x^2 + 2$ from $a = 0$ to $a = 1$ with $n = 4$ rectangles *i.e.* the upper bound.
c) Find the exact area.

Answer

a) $\frac{1}{4}(2+2\frac{1}{16}+2\frac{1}{4}+2\frac{9}{16})=2.21875$

b) $\frac{1}{4}(2\frac{1}{16}+2\frac{1}{4}+2\frac{9}{16}+3)=2.46875$

c) $\frac{7}{3}$

N.B. the answer to a) < c) < b) as you would expect.

Definition of the area under the curve

The area under the curve of a non-negative continuous function f over an interval $[a,b]$ is the limit of the sum of the n inscribed rectangles with equal base width, as n tends to infinity.

Area

$= \lim_{n\to\infty}((f(c_1)\Delta x + f(c_2)\Delta x + \ldots.. f(c_n)\Delta x)$

$= \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x$

Where $f(c_k)$ is the smallest value of f on the interval $[x_{k-1}, x_k]$ where $x_k = a + k\Delta x$ and

$\Delta x = \frac{b-a}{n}$

i.e. convergence, means $S_n \to L$ as $n \to \infty$

Just because the $\lim_{n\to\infty} a_n = 0$, it does not necessarily mean that the series will converge. But if the limit does not equal zero.

(as $S_n = \frac{a(1-r^n)}{1-r}$ $r \neq 1$ is the sum of a geometric series)

10.2.1 Convergence of infinite series

The infinite series $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \dots\dots + a_n + \dots\dots\dots$ **converges** with sum $L \Leftrightarrow$ the sequence of partial sums $\{s_n\}$ converges to L where S_n = $a_1+a_2+a_3\dots\dots a_n$.
If the series does not converge it is said to **diverge**.

a) So if the series has $-1 < r < 1$ it converges with $\lim_{n\to\infty} S_n = \frac{a}{1-r}$

b) if the series has $r = 1$ then it becomes
$a + a + a + a + a + \dots\dots\dots$
This series diverges as $s_n = an$ assuming $a \neq 0$

c) if the series has $r = -1$ then it becomes
$a - a + a - a + a - a + \dots\dots\dots$ assuming $a \neq 0$
This series diverges because the sequence of partial sums is a, 0, a, 0, a, 0,............ and this does not converge

Definition
The sum of a series is the limit of the sequence of its partial sums.
Bounded monotone series converge.

A bounded positive monotone series is a series where each sum is bigger than the previous one.

Consider the sequence of partial sums of a series
S_1, S_2, S_3 .....
If these are strictly monotonic increasing and bounded above then $S_n \to$ limit as $n \to \infty$ and all the $a_n > 0$. The series converges.

Partial fractions and telescoping series 10.2.2

For partial fractions see 10.6.1
Method of differences
If you get a fraction that can be split up into partial fractions, then you can telescope it – this works if the numerator is a constant

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} \equiv \sum_{r=1}^{n} \frac{A}{r} + \frac{B}{r+1} \qquad A(r+1) + Br \equiv 1 \qquad A = 1 \quad B = -1$$

$$\sum \frac{1}{r} + \frac{-1}{r+1}$$

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) \dots\dots\dots$$

$$\left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \frac{1}{1} - \frac{1}{n+1} = \frac{n}{n+1}$$

$$\Rightarrow \sum_{1}^{\infty} \frac{1}{r(r+1)} = \lim_{n\to\infty} \sum_{1}^{n} \frac{1}{r(r+1)} = 1$$

If you have three terms, the same process can be used, but in this case three parts will cancel out at a time. This works if the numerator is a linear function of r.

Example question

1. Show that $\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{1}{2}\left[\frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}\right]$

And $\sum_{r=1}^{\infty} \frac{1}{r(r+2)} = \frac{3}{4}$

2. Find $\sum_{r=1}^{n} \frac{3r+2}{r(r+1)(r+2)}$ and thus show that $\sum_{r=1}^{\infty} \frac{3r+2}{r(r+1)(r+2)} = 2$

Answer

2. $\frac{n(2n+3)}{(n+1)(n+2)}$

10.2.3 Tests for convergence

Comparison test

Sometimes integrals or series that cannot be summed easily can be compared with a simpler one and if the simpler one converges then the more complex one will converge.

Limit comparison test

An unknown series is compared term by term with a known one.

e.g. $S_n = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} \dots\dots$ where $x > 0$

is a complex series which we do not know how to sum. Does it even converge? So we compare it with the following series:
$S_n < 1 + x + x^2 + x^3 \dots\dots$

This is a geometric series with $r = x$ and so can be summed to infinity (if $x < 1$). Each term is clearly greater or equal than the corresponding term in S_n, so the series must also converge.

It doesn't tell us what happens for $x \geq 1$. We will see shortly that S_n diverges for $x \geq 1$.

Ratio test

For sequence $\{a_n\}$ if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ exists then

a) $L < 1 \Rightarrow$ series converges
b) $L > 1 \Rightarrow$ series diverges, or
c) $L = 1$ it could diverge or converge.

Questions

1) Does $\frac{1}{4}+\frac{1}{9}+\frac{1}{16}$...... converge? Justify your answer.

2) For what values of x does this series converge? And which series do you compare them with to show convergence?

$$\frac{x}{10}+\frac{x^2}{2\times 10^2}+\frac{x^3}{3\times 10^3}......$$

<u>Answers</u>

1) does converge

Compare the sequence with $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}...$

2) converges $-1 < x/10 < 1$ so that

$-10 < x < 10$; by comparison with $\frac{x}{10}+\frac{x^2}{10^2}+\frac{x^3}{10^3}......$

which is a geometric sequence.

Just because the limit of a function is zero, it does not necessarily mean that the series will converge. But if the limit does not equal zero, then the series will necessarily diverge.

Integral test

<u>Integral theorem</u>

If for $n \geq N$, $a_n = f(n)$ where $f(x) \geq 0$ and is a continuous, non-increasing function, then $\sum_{n=1}^{\infty} a_n$ and $\int_N^{\infty} f(x)\,dx$ either both converge or both diverge

10.2.4 The p-series

$\Sigma\frac{1}{n^p}$ is convergent for $p > 1$ and divergent otherwise. When $p = 1$ it is called the *harmonic series*

A p-series is of the form $\sum \frac{1}{n^r}$

The harmonic series is

$$\int_1^X \frac{1}{x}dx = \ln X \to \infty \; as \; X \to \infty$$

$\Rightarrow 1+\frac{1}{2}+\frac{1}{3}+.....\quad ....+\frac{1}{n}+....$ diverges by the integral test.

$1 + \underline{1/2} + \underline{1/3 + 1/4} + \underline{1/5 + 1/6 + 1/7 + 1/8} + \underline{1/9 + 1/10 + 1/11 + 1/12 + 1/13 + 1/14 + 1/15 + 1/16}$

Each underlined section is more than ½. You can always double the number of terms, so there can be an infinite number of halves and this can never tend to a limit and so can be shown to diverge independently..

<u>Worked example</u>

Show that $\sum_{r=1}^{N}\frac{1}{n^2}$ converges.

Find $\int_1^{\infty} 1/x^2 dx$

$= \lim_{X\to\infty}\int_1^X 1/x^2 dx$ if such a limit exists

$= \lim_{X\to\infty}[-1/x]_1^X$

$= \lim_{X\to\infty}[1-1/X] = 1$

N.B This can also be shown by the Comparison test as on the previous page.

<u>Worked Example</u>

Consider $\int_1^{\infty} f(x)dx$ where $f(x)=\frac{1}{x^n}$ when $n > 1$

All the functions $f(x)=\frac{1}{x^n}$ work in the same way when $n > 1$

$$\int_1^X \frac{1}{x^n}\,dx = \int_1^X x^{-n}\,dx$$

$$=\left[\frac{x^{-n+1}}{1-n}\right]_1^X = \left[\frac{X^{1-n}}{1-n}-\frac{1}{1-n}\right]$$

$$=0-\left(\frac{1}{1-n}\right) \therefore \int_1^{\infty}\frac{1}{x^n}\,dx = \lim_{X\to\infty}\left[\frac{X^{1-n}}{1-n}-\frac{1}{1-n}\right]$$

$$=\frac{1}{n-1}$$

As $n > 1$.

Use of integrals to estimate the sums of series

<u>Worked Example</u>

Estimate the sum of the square roots of n positive integers. $\int_0^1 \sqrt{x}\,dx = \lim_{n\to\infty}\frac{1}{n}\left[\sqrt{0}+\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+.......+\sqrt{\frac{n-1}{n}}\right]=\frac{2}{3}$ so by using the $\sum_0^{n-1}\sqrt{r}<\frac{2}{3}n\sqrt{n}<\sum_{r=1}^{n}\sqrt{r}$ definition of the area under the curve in 10.1.4 where $\Delta x=\frac{b-a}{n}=\frac{1-0}{n}$

So when n is large the sum is approximately equal to $\frac{2}{3}n\sqrt{n}$.

Question

Evaluate

$\lim_{n\to\infty} \frac{1^5+2^5+3^5+..........+n^5}{n^6}$ by showing that the limit is $\int_0^1 x^5\,dx$ and evaluating the integral.

Answer

$^1/_6$

10.3 Some common series

10.3.1 Series that converge absolutely and series that converge conditionally .

Consider the series $\sum_1^\infty a_n$ where a_n may be positive or negative. A necessary condition of convergence is $a_n \to 0$ as $n \to \infty$.

If $\sum_{n=1}^\infty a_n$ converges but $\sum_{n=1}^\infty |a_n|$ diverges, then the series conditionally converges.

If $\sum_{n=1}^\infty a_n$ converges but $\sum_{n=1}^\infty |a_n|$ converges, then the series absolutely converges.

10.3 2 Alternating sequences

An alternating sequence is one in which the signs alternate. It will contain $(-1)^n$.

Suppose {a_n

} is a sequence of positive terms which decreases monotonically to 0 as $n \to \infty$. Then

$\sum_1^n (-1)^r a_r$ as converges $n \to \infty$.

If { b_n } is an alternating sequence then $|b_n| \to 0$ monotonically is a sufficient condition for convergence.

Worked Example

Show that $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+....\sum_{n=1}\frac{(-1)^{n-1}}{n}$ converges to ln 2.

See the Taylor series in 10.5 to show that the limit is ln 2
This converges as (a) $1/n$ converges when n is positive and (b) decreases to 0 as $n \to \infty$ and (c) it is an alternating series.

"The absolute value of the truncation error is less than the magnitude of the next term in the series".

Examples

$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+........$ is divergent

$1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+........$ is conditionally convergent

$1-\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}+........$ is absolutely convergent

As $1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\frac{1}{4\sqrt{4}}+........$ converges as well.

$|x|<1$ which means $-1<x<1$

10.4 Power series

These are polynomials of infinite degree.

e.g. $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... + a_n x^n + ...$

As long as the series converges for a range of values of x. It will certainly converge for $x=0$ as all terms equal zero after a_0.

All the Maclaurins's and Taylor's series are examples of power series. See section 10.5.1

Power series can act in 3 ways:

i) They can converge at zero and diverge everywhere else.

ii) There is a positive number c for which they diverge when $|x| > c$ and converge absolutely for $|x| < c$. They may converge or diverge at c and $-c$.

iii) They can converge absolutely for all values of x.

Radius of convergence

The power series

$\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $-R < x < R$. where

$$R = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

R is called the *radius of convergence* (assuming that the limit exists)

Interval of convergence

Some power series in *h*, where $h = x - a$, have an interval of convergence for *x, a*

The interval of convergence is

$a - R < x < a + R$

so that the series converges of x in **an interval of convergence** centred at $x = a$ where a is the **centre of convergence**.

Determination of the radius of convergence by the ratio test

$\frac{x}{5} + \frac{x^2}{25} + \frac{x^3}{125}..........$ Ratio is $\frac{x}{5}$ therefore $\frac{a_n}{a_{n-1}}$ then $\frac{a_n}{a_{n+1}} = 5$ has radius of convergence 5

Question

Find the radius of convergence of

$\frac{x}{5} + \frac{x^2}{6} + \frac{x^3}{7}..........$

Answer

$\frac{a_n}{a_{n+1}} = \frac{n+4}{n+5}$

$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$

has radius of convergence 1

The ratio test is used because the difference between each term is less than $x + x^2 + x^3$ which converges but if $|x| = 1$ then it is the harmonic series which does not converge.

Question

Find the interval of convergence of

$\sum_{n=1}^{\infty} \frac{x^n}{n}$

Answer

Use the ratio test on the series of absolute values

$\lim_{n\to\infty}\left|\frac{x_{n+1}}{n+1} \times \frac{n}{x^n}\right| = |x|$

The original series converges if $|x| < 1$ and diverges if $|x| > 1$

When $x = 1$ the series becomes the harmonic series which diverges (shown already). When $x = -1$ the series becomes

$-(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}.......)$ which converges by alternating series condition..

So the series converges for $-1 \le x < 1$ which is the interval of convergence

---

10.5.1 Taylor polynomials and Maclaurin series

Taylor polynomials and series

Polynomials are very easy to differentiate, integrate and manipulate. Taylor found a way of finding a polynomial that gets closer and closer to a trigonometric or exponential function or others. The more terms you take of a Taylor polynomial the better the approximation.

Taylor's series is

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)(x-a)^2}{2}+\frac{f'''(a)(x-a)^3}{3!}\ldots+\frac{f^n(a)(x-a)^n}{n!}+\ldots..$$

or can be written with the *x = a + h to give*

$$f(a+h)=f(a)+f'(a)h+\frac{f''(a)h^2}{2}+\frac{f'''(a)h^3}{3!}\ldots\frac{f^n(a)h^n}{n!}$$

The Taylor approximation of degree 1 is a **linear approximation** to $f(x)$ near $x = a$. It is the tangent to $f(x)$ at $x = a$.

p(x)=f(a)+f'(a)(x–a)

The Taylor approximation of degree 2 is a **quadratic approximation** to $f(x)$ near $x = a$. It is the tangent parabola to $f(x)$ at $x = a$.

$$p(x)=f(a)+f'(a)(x-a)+\frac{f''(a)(x-a)^2}{2}$$

Worked example

1) Find the quadratic approximation for $\tan x$ about $x = \pi/3$. Use your answer to find a value for $\tan 1$.

$f(x)=\tan x$ $\qquad f\left(\frac{\pi}{3}\right)=\sqrt{3}$

$f'(x)=\sec^2 x$ $\qquad f'\left(\frac{\pi}{3}\right)=4$

$f''(x)=2\sec^2 x\tan x$ $\qquad f''\left(\frac{\pi}{3}\right)=2\times4\times\sqrt{3}=8\sqrt{3}$

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)(x-a)^2}{2}$$

$$=\sqrt{3}+4\left(x-\frac{\pi}{3}\right)+8\sqrt{3}\left(x-\frac{\pi}{3}\right)^2$$

$$=5.14-10.52x+6.93x^2$$

The first term polynomial would be $y=\sqrt{3}$

The linear polynomial would be $y=\left(\sqrt{3}-\frac{4\pi}{3}\right)+4x$

2) Draw each of these graphs one after the other and compare them with $\ln(1+x)$. It gives you a good idea how much difference each term in the series makes. See answer to (a) above

$y = x$

$y = x - \frac{x^2}{2}$

$y = x - \frac{x^2}{2} + \frac{x^3}{3}$

$y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$

$y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$

The error term

R_{n+1} is the remainder term or error term after $n + 1$ terms

$$R_{n+1}=\frac{h^{n+1}}{(n+1)!}f^{n+1}(a+\theta h)\quad\text{where}\quad 0\le\theta\le1\quad h=(x-a)$$

Question

Find the Taylor's series for $\ln(1+x)$ and the largest possible value for the error term for a polynomial of degree 4 with $-0.1\le x\le0.1$

Answer

$$x-\frac{1}{2}x^2+\frac{1}{3}x^3\ldots\ldots.(-1)^{n+1}\frac{x^n}{n}$$

The error term is $\frac{x^5}{5(1+\theta x)^5}$ when using a polynomial of degree 4 .

The magnitude of this is maximized when $x = -0.1$ and $\theta = 1$ to make the denominator as small as possible and the numerator as large as possible

$$\frac{1}{5}\frac{10^{-5}}{0.9^5}=3.39\times10^{-6}$$

10.5.2 Maclaurins's series

These are given to you in the formula book but it is important to see how they work. They are just Taylor's series around the point x=0.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\ldots \frac{x^n}{n!} \ldots\ldots \quad \forall x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots\ldots \quad \forall x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots\ldots \quad \forall x$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \ldots\ldots \quad -1 < x \le 1$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \ldots\ldots \quad -1 < x \le 1$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \ldots \quad |x| < 1$$

if n is not a positive integer.
You may recognize the last one as the binomial theorem. If n is a positive integer the series terminates at x^n.
You can use these by substituting in the corresponding value of x.

Worked example

Find e^2 to 3d.p.
First choose the appropriate series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\ldots + \frac{x^n}{n!} \ldots$$

then substitute the value of x

$$e^x = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \ldots\ldots\ldots + \frac{2^9}{9!} \ldots\ldots$$

$$e^2 \approx 7.39$$

Check that $R_{10} < \frac{z^{10}}{10!}(e^z)$

Question

a. Use a Maclaurin's series to find cos (0.2) to 2 sig. fig.
b. Find the Maclaurin series up to the x^3 of e^{2x}
(hint - use the e^x series and replace each x by $2x$)
c. Find the Taylor's series from first principles by finding $f'(x)$, $f''(x)$ and, $f'''(x)$ for $f(x) = (1+x)^{1/2}$. Then find the Taylor series for $f(3+h)$. Find for which values it's valid and write down the Taylor's series up to the h^3 term.
Use the answer to form a series for $f(x) = (1+2x)^{1/2}$

Answer

a. 0.98

b. $1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!}$

c. i) $\frac{1}{2}(1+x)^{-1/2}$ ii) $-1/4(1+x)^{-3/2}$ iii) $3/8(1+x)^{-5/2}$
iv) $-15/16(1+x)^{-7/2}$ so substituting $x = 3$ the series is

$$2 + \frac{h}{4} - \frac{h^2}{64} + \frac{1}{512}h^3$$

10.5.3 L'Hopital's rule

Suppose that $f(a) = g(a) = 0$ and that $f'(a)$ and $g'(a)$ exist and that $g'(a)$ is not equal to zero

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \lim_{x\to a}\left[\frac{f'(x)}{g'(x)}\right]$$

Worked Example

Find

N.B. Note the use of L'Hopital's rule rather than the quotient rule.

$$\lim_{x\to 0}\frac{e^x(1-\cos x)}{\sin^2 x} \qquad \text{Let } f(x) = e^x(1-\cos x)$$

$$g(x) = \sin^2 x$$

$$\lim_{x\to 0}\frac{e^x\sin x + e^x(1-\cos x)}{2\sin x\cos x} \qquad f'(x) = e^x(1-\cos x) + e^x\sin x$$

$$g'(x) = 2\sin x\cos x = \sin 2x$$

This is still $\frac{0}{0}$ which isn't defined so differentiate again.

$$\lim_{x\to 0}\frac{e^x\cos x + e^x - e^x\cos x + e^x\sin x}{2\sin x\cos x} \qquad f''(x) = e^x(1-\cos x) + e^x\sin x + e^x\sin x + e^x\cos x$$

$$g''(x) = 2\cos 2x$$

$$\lim_{x\to 0}\frac{e^x + e^x\sin x}{2\cos 2x} = 1/2 \qquad e^x = 1 \text{ as } x\to 0$$

$$\cos 2x = 1 \text{ as } x\to 0$$

Question

Find the limit using L'Hopital's Rule of

$$\lim_{x\to e}\frac{1-\ln x}{\sin(x-e)}$$

Answer

$$\lim_{x\to e}\frac{1-\ln x}{\sin(x-e)} = \lim_{x\to e}\frac{-\frac{1}{x}}{\cos(x-e)} = -\frac{\frac{1}{e}}{1} = -\frac{1}{e}$$

10.6.1 Differential equations

A differential equation is an equation containing a derivative of one variable differentiated with respect to another *e.g.* $\frac{dy}{dx}$ or a higher derivative e.g. $\frac{d^3x}{dy^3}$. You are trying to find what y is in terms of x when you are solving a differential equation (or an implicit relationship).

First order differential equation

A first order differential equation is one that contains $\frac{dy}{dx}$ and no higher derivative

While a second order differential equation is one that contains $\frac{d^2y}{dx^2}$ and no higher derivatives.

You only need to learn about first order differential equations. Remember all the types of integration that you have learned.

Integration by substitution you want to substitute a function that when you differentiate will simplify the original function and make it integrable.

$\int xe^{x^2}dx$	$u = x^2$
	$\frac{du}{dx} = 2x$
$\int xe^{u}\frac{du}{2x}$	$\frac{du}{2x} = dx$ $or\ du = 2xdx$
$\int \frac{1}{2}e^{u}du$	
$\frac{1}{2}e^{u} + c = \frac{1}{2}e^{x^2} + c$	

Integration by parts.

Often the function *f(x)* = *xg(x)*. You then let *x* = *u* and *g(x)dx* = *dv*.

$$\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$$

$\int xe^{x}dx$	$u = x$ $\frac{du}{dx} = 1$	$\frac{dv}{dx} = e^{x}$ $v = e^{x}$
$xe^{x} - \int e^{x}dx$		
$xe^{x} - e^{x} + c$		

One notable exception is that when you have an ln x term it then usually becomes the u term.

$\int x\ln x\,dx$	$u = \ln x$	$dv = x$ $v = \frac{1}{2}x^2$
$= \frac{1}{2}x^2\ln x - \int \frac{1}{2}x^2\frac{1}{x}dx$	$\frac{du}{dx} = \frac{1}{x}$	
$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + c$		

Standard integrals

$\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c$

$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$

$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln(ax+b) + c$

$\int \sin(ax+b) = -\frac{1}{a}\cos(ax+b) + c$

$\int \cos(ax+b) = \frac{1}{a}\sin(ax+b) + c$

Questions

1) $\int e^{3x}\,dx$

2) $\int \frac{1}{3x-2}dx$

3) $\int \sin 4x\,dx$

4) $\int \cos(3x-1.8)dx$

Answers

1) $\frac{1}{3}e^{3x} + c$

2) $\frac{1}{3}\ln|3x-2| + c$

3) $-\frac{1}{4}\cos 4x + c$

4) $\frac{1}{3}\sin(3x - 1.8) + c$

Partial fractions

It is easy to integrate something of the form

$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + c$

But if you get something like $\frac{x}{(x+1)(x+2)}$

It is necessary to separate it using partial fractions. You are doing the opposite of adding fractions. Separate the denominator into linear fractions if possible and put constant terms on each one *.e.g.*

$$\int \frac{x}{(x+1)(x+2)}dx = \frac{A}{x+1} + \frac{B}{x+2} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$$

$So\ A(x+2)+B(x+1)=x$

$If\ x=-1$ then $A=-1$

$If\ x=-2$ then $B=2$

$$= \int \frac{-1}{x+1} + \frac{2}{x+2} = -\ln|x+1| + 2\ln|x+2| + c = \ln\left|\frac{A(x+2)^2}{x+1}\right|$$

You have already covered equations where you can separate the variables in the core.

You are attempting to keep the dy multiplied on the right side with all the terms containing *y*, and the dx multiplied by all the x terms of the equation.

Questions

Find general solutions to

1. $\frac{dy}{dx} - e^x y^5 = 0$
2. $x^3 \frac{dy}{dx} = 6y^3$
3. $\frac{dy}{dx} + 4x \operatorname{cosec} y = 0$
4. $\frac{dy}{dx} = \frac{3y}{2x+1}$
5. $\frac{dy}{dx} = 3x^3(1+y^2)$

Find particular solutions to

6. $(\cos y)\frac{dy}{dx} = x^3 \operatorname{cosec}^3 y$; when $x = 2$, $y = \frac{\pi}{2}$
7. $3x\frac{dy}{dx} + 9 = y^2$; when $x = \frac{1}{3}$, $y = -2$

Answers

1) $-\frac{1}{4} y^{-4} = e^x + c$
2) $-1/2y^{-2} = -3x^{-2} + c$
3) $\cos y = 2x^2 + c$
4) $y = A(2x+1)^{3/2}$
5) $y = \tan(3/4\ x^4 + c)$
6) $\frac{1}{4}\sin^4 y = \frac{1}{4}x^4 - 3.75$
7) $\left(\frac{y-3}{y+3}\right) = -45x^2 \Rightarrow y = \frac{3(1-45x^2)}{1+45x^2}$

Question

a) Find the general equation of the family of curves for which the gradient at any point on the curve is the same as the y-coordinate at the point.

b)

By substituting $x=X-1$ and $y=Y+3$ reduce the differential equation $\frac{dy}{dx} = \frac{4x-y+7}{2x+y-1}$ (you do not need to solve it)

Answer

a) $y = A e^x$

b) $\frac{dY}{dX} = \frac{4X-Y}{2X+Y}$

It is now a homogeneous equation that can be solved with the substitution $y = vx$

Solving homogeneous equations using the substitution $y = vx$,

In this context, homogeneous equations are equations where in each term the powers of *x's* and *y's* all multiply to give the same power

Worked example

Solve $\frac{dy}{dx} = \frac{x+2y}{x}$ by making the substitution $y = vx$ where v is a function of x, finding the particular solution when $x = 1$ and $y = 2$

When $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Differentiate *vx* by using the product rule remembering that the differential of *v* will be $\frac{dv}{dx}$

Substituting this into the original equation

$$v + x\frac{dv}{dx} = \frac{x+2y}{x}$$

$$v + x\frac{dv}{dx} = 1 + 2v$$

$$x\frac{dv}{dx} = 1 + v$$

The variables can be separated now.

$$\int \frac{1}{v+1}\,dv = \int \frac{1}{x}\,dx$$

$$\ln|v+1| = \ln|x| + c$$

$$\ln|v+1| = \ln|Ax|$$

Now substituting $y = 2$ and $x = 1$ gives $A = 3$.

$v + 1 = Ax$ as $v = \frac{y}{x}$

so $y = Ax^2 - x$

So the particular solution is $y = 3x^2 - x$

Question

Solve $x\frac{dy}{dx} = 3x - y$

When $x = 1$ and $y = 2$.

Answer

$$y = \frac{3x^2 + 1}{2x}$$

Slope fields and their geometric equations

To sketch the slope field find the gradient of every co-ordinate point on a grid, by substituting the value of x and y in the differential equation. Then sketch the gradient at each point. Then join the curves up.

Question

Fill in the gradients when $\frac{dy}{dx} = \frac{x}{y}$ on a grid with $-5 \le x \le 5$; $-5 \le y \le 5$

Sketch 3 of the possible solutions to the differential equation.

Numerical solution using Euler's method.

Euler's method is based on the approximation $y_1 \approx y_0 + h\ dy/dx$ its accuracy depends on h being small. Set your answers out as a table. In an exam you will probably be asked to fill in missing numbers or only do the first steps.

Worked Example

$\frac{dy}{dx} = x - y$ and $y = 2$ when $x = 0$. Find y when $x = 0.3$. Taking x step as $h = 0.1$.

Step	x	y	$y_{n+1} = y_n + h dy/dx$
0	0	2	2 + 0.1(0 – 2)
1	0.1	1.8	1.8 + 0.1 (0.1 – 1.8)
2	0.2	1.63	1.63 + 0.1 (0.2 – 1.63)=1.487
3	0.3	1.487	

Question

Use Euler's method to solve $\frac{dy}{dx} = x - y^3$ when $y = 0$ and $x = 0$ and $h = 0.1$ find y when $x = 0.3$.

Answer

When $x = 0.3$ $y = 0.0299$

Functions of the form $\frac{dy}{dx} + ky = Q(x)$

Firstly consider $\frac{dy}{dx} + ky = 0$

$\Rightarrow y = Ae^{-kx}$ from separating variables

This gives L.H.S = 0

Ae^{-kx} is called the complementary function.(C.F.)

Then find a function " by inspection" that satisfies

$\frac{dy}{dx} + ky = Q(x)$

This is called the particular integral (P.I.)

The general solution is then given by

y = C.F. + P.I.

Worked example

Solve $\frac{dy}{dx} + 4y = 3x$

The homogeneous equation is $\frac{dy}{dx} + 4y = 0$

Which separates to $\int \frac{1}{y}\,dz = -\int 4\,dx$

$\ln y = -4x + c$

$y = Ae^{-4x}$ which by inspection is the complementary function.

To find P.I. let $y = ax + b$

$\frac{dy}{dx} = a$

$\frac{dy}{dx} + 4y = 3x$

$\Rightarrow a + 4(ax + b) = 3x$

$4ax + (a + 4b) = 3b$

$a = \frac{3}{4}$ $b = \frac{-3}{16}$

So P.I. = $\frac{3}{4}x - \frac{3}{16}$

So the general solution is

$$y = Ae^{-4x} + \frac{3}{4}x - \frac{3}{16}$$

Then find A if the boundary conditions are given

If $Q(x)$ is a polynomial the particular solution will be a polynomial with the same degree

If $Q(x)$ is an exponential the particular solution will be Ae^{bx}

If $Q(x)$ is sine or cosine function the particular solution will a combination of $A\sin bx$ and $B\cos bx$

Questions

1) $\frac{dy}{dx}+3y=2,\ y=1$ when $x=0$

This can be done as separable variables as well as C.F. and P.I. method.

2) $\frac{dx}{dt}-2x=4e^{t},x=3$ when $t=0$

3) $\frac{dy}{dx}+\frac{1}{3}y=4x+6,y=2$ when $x=0$

4) $\frac{dx}{dt}+3x=4\sin t-7\cos t$

5) $\frac{dy}{dx}+3y=2e^{-3x}$

Hint take $y = Axe^{-3x}$ as P.I.

Answers

1. $y=\frac{2}{3}+\frac{1}{3}e^{-3x}$ 2. $y = 7e^{2t} - 4e^{t}$ 3. $y = 12x - 18 + 20e^{-\frac{1}{3}x}$ 4. $x=Ae^{-3t}+\frac{1}{2}\sin t-\frac{5}{2}\cos t$ 5. $y = 2xe^{-3x} + Ae^{-3x}$

Integrating Factor

$\frac{dy}{dx}+yP(x)=Q(x)$

The equation is multiplied by the integrating factor

$e^{\int P(x)dx}=R(x)$

$\Rightarrow \frac{dy}{dx}e^{\int P(x)dx}+yP(x)e^{\int P(x)dx}=Q(x)e^{\int P(x)dx}$

The L.H.S. simplifies to

$\frac{d}{dx}(ye^{\int P(x)dx})$ so equation becomes

$\frac{d}{dx}\left(ye^{P(x)dx}\right)=\int Q(x)e^{P(x)dx}dx$

This looks complicated but the examples will make it clearer.

Example

Use integration by parts twice to get the last line

$\frac{dy}{dx}+2xy=x^3$

$\text{I.F.}=e^{\int 2x\,dx}=e^{x^2}$

$e^{x^2}\frac{dy}{dx}+2xe^{x^2}y=x^3e^{x^2}\Rightarrow\frac{d}{dx}(ye^{x^2})=x^3e^{x^2}$

$ye^{x^2}=\int x^3e^{x^2}dx$

$=\frac{1}{2}(x^2-1)e^{x^2}+c$

$=\frac{1}{2}(x^2-1)e^{x^2}+Ce^{-x^2}$

C.F. – complementary function

P.I. – particular integral

Note that all the C.F/ P.I. questions can be solved by using the integrating factor, (try them as an exercise), but as you can see in the worked example it can often lead to a tricky integral to be solved on the R.H.S.

Questions

1. $\frac{dy}{dx}+\frac{3y}{x}=x$

2. $x^2\frac{dy}{dx}-2xy=e^{x}$

3. $\frac{dy}{dx}-2xy=e^{x^2}$

Answers

1. $\frac{1}{5}x^2+Ax^{-3}$ Hint $\text{I.F.}=e^{\ln(x^3)}=x^3$

2. Hint L.H.S is already $\frac{d}{dx}(x^2y)$ $\quad Ans\ y=\frac{e^{x}+A}{x^2}$

3. $y=e^{x^2}(x+c)$

Now get hold of some past paper questions, bearing in mind the syllabus has changed in papers before 2005. Look for series and limit questions in the old calculus questions.

You should certainly do at least one past paper under timed conditions before the exam.

Learn the definitions and all the tests. They are easy marks.

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