10.3 Computing Limits Algebraically

Question 1: How can a limit be computed algebraically?

Question 2: How do you evaluate limits involving difference quotients?

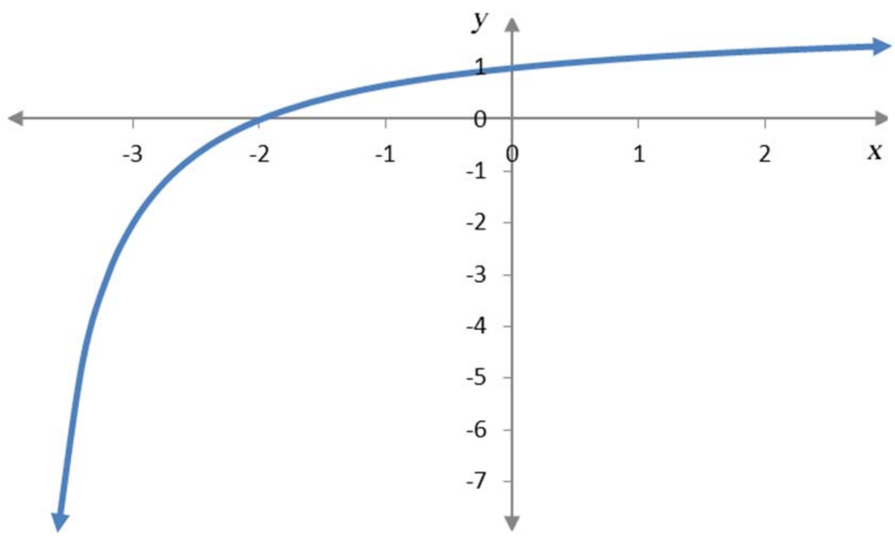
In section 10.1, we examined tables and graphs to help us evaluate limits. For instance, to evaluate the limit

$$\lim_{x \to 0} \frac{2x+4}{x+4}$$

we could construct a table of approximate y values near $x = 0$.

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$\frac{2x+4}{x+4}$	0.9744	0.9974	0.9997	1	1.0002	1.0025	1.0244

We could also graph the function to get an idea of how the y values behave as x gets closer and closer to 0.



For either representation, as x get closer and closer to 0, the corresponding y value gets closer and closer to 1. This means the value of the limit is 1,

$$\lim_{x \to 0} \frac{2x+4}{x+4} = 1$$

If you look at the table or the graph, you'll notice that the y value for the expression $\frac{2x+4}{x+4}$ is 1 at the value $x=0$. This suggests that you could simply substitute $x=0$ into the expression to evaluate the limit. By doing this, we can avoid creating the table or making the graph. In this section, we'll examine the situations when this strategy works and ways to work around the problems that occur when it is not possible.

Question 1: How can a limit be computed algebraically?

For many functions, we can evaluate the limit by substituting the value x is approaching into the limit's expression.

> **Basic Rule for Evaluating Limits Algebraically**
>
> If a function $f(x)$ is made up of additions, subtractions, multiplications, divisions, powers and roots, then the limit as x approaches a may be evaluated by substituting $x=a$ into the function $f(x)$ as long as this function value is defined.

The basic rule applies to one and two sided limits.

Example 1 Evaluate the Limit

Evaluate each of the limits below.

a. $\lim_{x\to 2}\left(x^2-2x+7\right)$

Solution The polynomial is made up of additions, subtractions, powers, and multiplications so we may set $x=2$ to find the limit:

$$\begin{aligned}\lim_{x\to 2}\left(x^2-2x+7\right) &= 2^2-2\left(2\right)+7\\ &= 7\end{aligned}$$

b. $\lim_{t\to -1}\frac{2t+4}{t-1}$

Solution The rational expression is made up of additions, subtractions, mulitplication and division so substitute $t=-1$ to find the limit:

$$\lim_{t \to -1} \frac{2t+4}{t-1} = \frac{2(-1)+4}{-1-1}$$
$$= -1$$

c. $\lim_{x \to 4} \frac{\sqrt{x}}{x-3}$

Solution As with the earlier parts, substitute the value x is approaching to compute the limit:

$$\lim_{x \to 4} \frac{\sqrt{x}}{x-3} = \frac{\sqrt{4}}{4-3}$$
$$= 2$$

d. $\lim_{x \to 2} \frac{x^2-4}{x-2}$

Solution When we substitute $x = 2$ into the rational expression, the denominator is undefined. This means we need an alternate strategy for evlauating this limit.

The basic rule works in many cases. But limits like the one in part d of Example 1 require some extra steps. In this limit, substituting the value x is approaching in the limit results in

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = \frac{2^2-4}{2-2} = \frac{0}{0}$$

The numerator and denominator are both equal to zero. This occurs because the numerator contains a hidden factor of $x-2$,

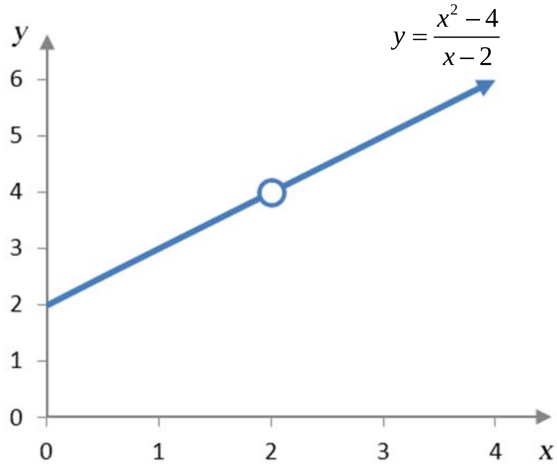
$$\frac{x^2-4}{x-2} = \frac{(x+2)(x-2)}{x-2}$$

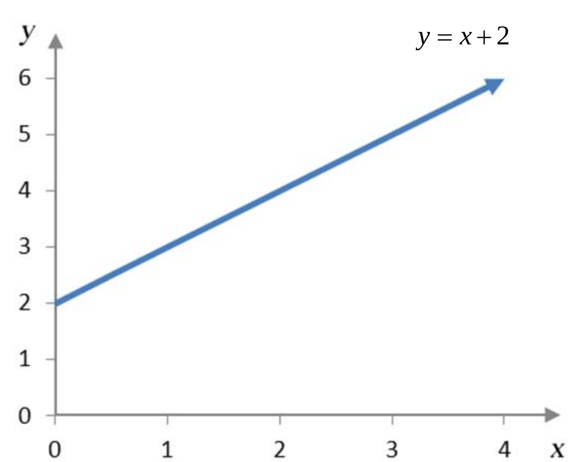
The factors of $x-2$ in the numerator and denominator lead to the zeros in the numerator and denominator. Since these factors are the same, we can simplify the rational expression to yield

$$\frac{x^2-4}{x-2}=\frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}}$$

$$=x+2$$

We have written $\frac{x^2-4}{x-2}$ and $x+2$ as equal, but in what sense? Let's examine the graph of each expression.





The graphs are identical except for the point at $x=2$. The graph on the left is not defined at $x=2$, but the graph on the right is defined there. This has no effect on the limit as x approaches 2. The y values on both functions get closer and closer to 4 as x approaches 2. This means we can simplify the rational expression and then substitute $x=2$ into the result. This yields the limit

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x-2}$$
$$= \lim_{x \to 2} (x+2)$$
$$= 2 + 2$$
$$= 4$$

For limits that give $\frac{0}{0}$ when the value is substituted, simplifying the expression often allows the limit to be evaluated by substitution.

Example 2 Evaluate the Limit

Evaluate each of the limits below.

a. $\lim_{x \to 3} \frac{x^2 + x - 12}{x - 3}$

Solution The numerator and denominator are both zero at $x = 3$. To evaluate the limit, simplify the expression before making the substitution:

$$\lim_{x \to 3} \frac{x^2 + x - 12}{x - 3} = \lim_{x \to 3} \frac{(x+4)(x-3)}{x-3} \quad \text{Factor numerator}$$
$$= \lim_{x \to 3} (x+4) \quad \text{Simplify}$$
$$= 7 \quad \text{Substitute } x = 3$$

b. $\lim_{x \to 0^+} \frac{\frac{1}{x+5} - \frac{1}{5}}{x}$

Solution When you set $x = 0$, the numerator and denominator are both zero. To simplify the expression, combine the fractions in the numerator.

$$\lim_{x\to 0^+}\frac{\frac{1}{x+5}-\frac{1}{5}}{x}=\lim_{x\to 0^+}\frac{\frac{5}{5(x+5)}-\frac{x+5}{5(x+5)}}{x}$$ Get a common denominator of $5(x+5)$

$$=\lim_{x\to 0^+}\frac{\frac{-x}{5(x+5)}}{x}$$ Combine fractions together

$$=\lim_{x\to 0^+}\frac{-x}{5(x+5)}\cdot\frac{1}{x}$$ Dividing by x is the same as multiplying by its reciprocal

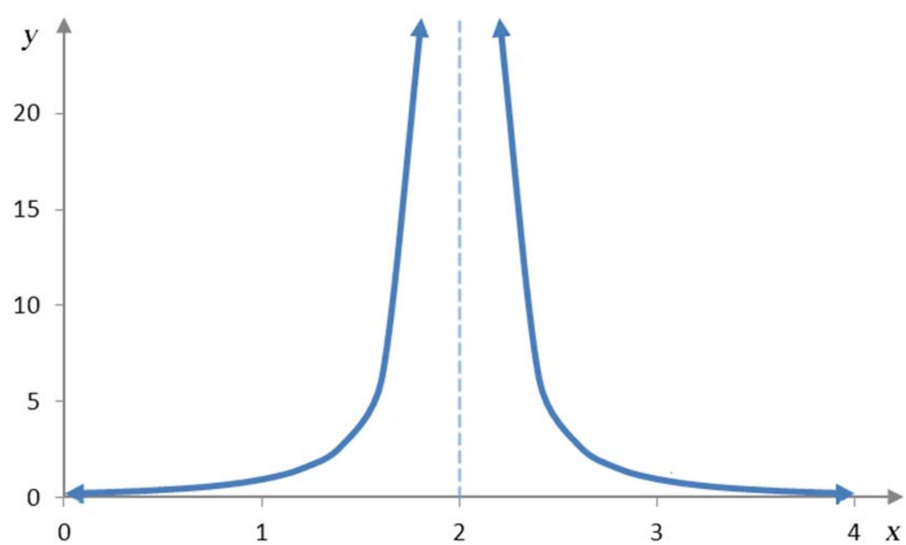
$$=\lim_{x\to 0^+}\frac{-1}{5(x+5)}$$ Simplify

$$=-\frac{1}{25}$$ Substitute $x=0$

If the expression is undefined, but not because both the numerator and denominator are zero, we fall back to a table or graph to evaluate the limit. For example, in the limit

$$\lim_{x\to 2}\frac{1}{(x-2)^2}$$

the denominator is zero when $x=2$ is substituted. However, the numerator is not equal to zero. In this situation, a graph allows us to examine the behavior near $x=2$.



As x gets closer and closer to x from the left or right, the y values grow larger and larger. In terms of the limit, we write

$$\lim_{x \to 2} \frac{1}{\left(x-2\right)^2} = \infty$$

This means the limit does not exist since the y values grow larger and larger.

Question 2: How do you evaluate limits involving difference quotients?

In calculus, we frequently encounter expressions of the form

$$\frac{f(a+h)-f(a)}{h}$$

This type of expression is called a difference quotient. It may appear as part of a limit,

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

If we try to evaluate this limit by setting $h=0$, both the numerator and denominator are zero. This indicates that we'll need to try to simplify the difference quotient to evaluate the limit.

Example 3 Evaluate the Limit

Evaluate the limit

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

for $f(x)=5x-1$ and $a=2$.

Solution Before attacking the limit, write out the difference quotient with the function and constant a. The two function values in the numerator are

$$f(2)=5(2)-1=9$$

$$f(2+h)=5(2+h)-1=9+5h$$

With these values, we simplify the difference quotient:

$$\begin{aligned}\frac{f(2+h)-f(2)}{h} &= \frac{9+5h-9}{h} \\ &= \frac{5h}{h} \\ &= 5\end{aligned}$$

With this simplification, the limit becomes

$$\begin{aligned}\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0} 5 \\ &= 5\end{aligned}$$

Since there is no h in the simplified difference quotient, setting $h=0$ has no effect on the constant. The limit of the constant is the equal to the constant.

As the function becomes more complicated, the algebra required to simplify the difference quotient may be more complicated. Pay careful attention to negative signs and removing parentheses.

Example 4 Evaluate the Limit

Evaluate the limit

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

for $f(x)=x^2-2x$ and $a=1$.

Solution The function value $f(1)$ is easy to find,

$$f(1)=1^2-2(1)=-1$$

However, the other function value requires several steps to simplify:

$$
\begin{aligned}
f(1+h) &= (1+h)^2 - 2(1+h) \\
&= 1 + 2h + h^2 - 2(1+h) \\
&= 1 + 2h + h^2 - 2 - 2h \\
&= h^2 - 1
\end{aligned}
$$

Put these values into the difference quotient and simplify to yield,

$$
\begin{aligned}
\frac{f(1+h) - f(1)}{h} &= \frac{h^2 - 1 - (-1)}{h} \\
&= \frac{h^2}{h} \\
&= h
\end{aligned}
$$

With the difference quotient simplified, we can evaluate the limit:

$$
\begin{aligned}
\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0} h \\
&= 0
\end{aligned}
$$

■

We'll evaluate limits involving difference quotients in Chapter 11. The difficulty in evaluating the limits is not the concept of the limit itself. Instead, the algebra required to simplify the difference quotient is the most challenging aspect. It will require careful attention to algebra details.