Chapter 4

Series

> Divergent series are the devil, and it is a shame to base on them any demonstration whatsoever. (Niels Henrik Abel, 1826)
>
> This series is divergent, therefore we may be able to do something with it. (Oliver Heaviside, quoted by Kline)

In this chapter, we apply our results for sequences to series, or infinite sums. The convergence and sum of an infinite series is defined in terms of its sequence of finite partial sums.

4.1. Convergence of series

A finite sum of real numbers is well-defined by the algebraic properties of $\mathbb{R}$, but in order to make sense of an infinite series, we need to consider its convergence. We say that a series converges if its sequence of partial sums converges, and in that case we define the sum of the series to be the limit of its partial sums.

Definition 4.1. Let (a_n) be a sequence of real numbers. The series

$$\sum_{n=1}^{\infty} a_n$$

converges to a sum $S \in \mathbb{R}$ if the sequence (S_n) of partial sums

$$S_n = \sum_{k=1}^{n} a_k$$

converges to S as $n \to \infty$. Otherwise, the series diverges.

If a series converges to S, we write

$$S = \sum_{n=1}^{\infty} a_n.$$

We also say a series diverges to $\pm\infty$ if its sequence of partial sums does. As for sequences, we may start a series at other values of n than $n = 1$ without changing its convergence properties. It is sometimes convenient to omit the limits on a series when they aren't important, and write it as $\sum a_n$.

Example 4.2. If $|a| < 1$, then the geometric series with ratio a converges and its sum is

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}.$$

This series is simple enough that we can compute its partial sums explicitly,

$$S_n = \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}.$$

As shown in Proposition 3.31, if $|a| < 1$, then $a^n \to 0$ as $n \to \infty$, so that $S_n \to 1/(1-a)$, which proves the result.

The geometric series diverges to ∞ if $a \geq 1$, and diverges in an oscillatory fashion if $a \leq -1$. The following examples consider the cases $a = \pm 1$ in more detail.

Example 4.3. The series

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + \dots$$

diverges to ∞, since its nth partial sum is $S_n = n$.

Example 4.4. The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + \dots$$

diverges, since its partial sums

$$S_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

oscillate between 0 and 1.

This series illustrates the dangers of blindly applying algebraic rules for finite sums to series. For example, one might argue that

$$S = (1-1) + (1-1) + (1-1) + \dots = 0 + 0 + 0 + \dots = 0,$$

or that

$$S = 1 + (-1+1) + (-1+1) + \dots = 1 + 0 + 0 + \dots = 1,$$

or that

$$1 - S = 1 - (1 - 1 + 1 - 1 + \dots) = 1 - 1 + 1 - 1 + 1 - \dots = S,$$

so $2S = 1$ or $S = 1/2$. The Italian mathematician and priest Luigi Grandi (1710) suggested that these results were evidence in favor of the existence of God, since they showed that it was possible to create something out of nothing.

Telescoping series form another class of series whose partial sums can be computed explicitly and then used to study their convergence. We'll give one example.

Example 4.5. The series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \dots$$

converges to 1. To show this, we observe that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

so

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)\\ &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{n} - \frac{1}{n+1}\\ &= 1 - \frac{1}{n+1},\end{aligned}$$

and it follows that

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1.$$

A condition for the convergence of series with positive terms follows immediately from the condition for the convergence of monotone sequences.

Proposition 4.6. A series $\sum a_n$ with positive terms $a_n \geq 0$ converges if and only if its partial sums

$$\sum_{k=1}^{n} a_k \leq M$$

are bounded from above, otherwise it diverges to ∞.

Proof. The partial sums $S_n = \sum_{k=1}^{n} a_k$ of such a series form a monotone increasing sequence, and the result follows immediately from Theorem 3.29 □

Although we have only defined sums of convergent series, divergent series are not necessarily meaningless. For example, the Cesàro sum C of a series $\sum a_n$ is defined by

$$C = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} S_n, \qquad S_n = a_1 + a_2 + \dots + a_n.$$

That is, we average the first n partial sums the series, and let $n \to \infty$. One can prove that if a series converges to S, then its Cesàro sum exists and is equal to S, but a series may be Cesàro summable even if it is divergent.

Example 4.7. For the series $\sum(-1)^{n+1}$ in Example 4.4, we find that

$$\frac{1}{n} \sum_{k=1}^{n} S_k = \begin{cases} 1/2 + 1/(2n) & \text{if } n \text{ is odd,}\\ 1/2 & \text{if } n \text{ is even,}\end{cases}$$

since the S_n's alternate between 0 and 1. It follows the Cesàro sum of the series is $C = 1/2$. This is, in fact, what Grandi believed to be the "true" sum of the series.

Cesàro summation is important in the theory of Fourier series. There are many other ways to sum a divergent series or assign a meaning to it (for example, as an asymptotic series), but we won't discuss them further here, and we'll only consider the sum of a series to be defined if the series converges.

4.2. The Cauchy condition

The following Cauchy condition for the convergence of series is an immediate consequence of the Cauchy condition for the sequence of partial sums.

Theorem 4.8 (Cauchy condition)**.** The series

$$\sum_{n=1}^{\infty} a_n$$

converges if and only for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left|\sum_{k=m+1}^{n} a_k\right| = |a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon \qquad \text{for all } n > m > N.$$

Proof. The series converges if and only if the sequence (S_n) of partial sums is Cauchy, meaning that for every $\epsilon > 0$ there exists N such that

$$|S_n - S_m| = \left|\sum_{k=m+1}^{n} a_k\right| < \epsilon \qquad \text{for all } n > m > N,$$

which proves the result. □

A special case of this theorem is a necessary condition for the convergence of a series, namely that its terms approach zero. This condition is the first thing to check when considering whether or not a given series converges.

Theorem 4.9. If the series

$$\sum_{n=1}^{\infty} a_n$$

converges, then

$$\lim_{n\to\infty} a_n = 0.$$

Proof. If the series converges, then it is Cauchy. Taking $m = n - 1$ in the Cauchy condition in Theorem 4.8, we find that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n| < \epsilon$ for all $n > N$, which proves that $a_n \to 0$ as $n \to \infty$. □

Example 4.10. The geometric series $\sum a^n$ converges if $|a| < 1$ and in that case $a^n \to 0$ as $n \to \infty$. If $|a| \geq 1$, then $a^n \not\to 0$ as $n \to \infty$, which implies that the series diverges.

The condition that the terms of a series approach zero is not, however, sufficient to imply convergence. The following series is a fundamental example.

Example 4.11. The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

diverges, even though $1/n \to 0$ as $n \to \infty$. To see this, we collect the terms in successive groups of powers of two,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{16}\right) + \dots \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \frac{1}{16} + \dots + \frac{1}{16}\right) + \dots \\ &> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots .\end{aligned}$$

In general, for every $n \geq 1$, we have

$$\begin{aligned}\sum_{k=1}^{2^{n+1}} \frac{1}{k} &= 1 + \frac{1}{2} + \sum_{j=1}^{n} \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{k} \\ &> 1 + \frac{1}{2} + \sum_{j=1}^{n} \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{2^{j+1}} \\ &> 1 + \frac{1}{2} + \sum_{j=1}^{n} \frac{1}{2} \\ &> \frac{n}{2} + \frac{3}{2},\end{aligned}$$

so the series diverges. We can similarly obtain an upper bound for the partial sums,

$$\sum_{k=1}^{2^{n+1}} \frac{1}{k} < 1 + \frac{1}{2} + \sum_{j=1}^{n} \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{2^j} < n + \frac{3}{2}.$$

These inequalities are rather crude, but they show that the series diverges at a logarithmic rate, since the sum of 2^n terms is of the order n. A more refined argument, using integration, shows that

$$\lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \log n\right] = \gamma$$

where $\gamma \approx 0.5772$ is the Euler constant. (See Example 12.43.) This rate of divergence is very slow. It takes 12367 terms for the partial sums of harmonic series to exceed 10, and more than 1.5×10^{43} terms for the partial sums to exceed 100.

4.3. Absolutely convergent series

There is an important distinction between absolutely and conditionally convergent series.

Definition 4.12. The series

$$\sum_{n=1}^{\infty} a_n$$

converges absolutely if

$$\sum_{n=1}^{\infty} |a_n| \text{ converges,}$$

and converges conditionally if

$$\sum_{n=1}^{\infty} a_n \text{ converges, but } \sum_{n=1}^{\infty} |a_n| \text{ diverges.}$$

As we show in Proposition 4.17 below, every absolutely convergent series converges. For series with positive terms, there is no difference between convergence and absolute convergence. Also note from Proposition 4.6 that $\sum a_n$ converges absolutely if and only if the partial sums $\sum_{k=1}^{n} |a_k|$ are bounded from above.

Example 4.13. The geometric series $\sum a^n$ is absolutely convergent if $|a| < 1$.

Example 4.14. The alternating harmonic series,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

is not absolutely convergent since, as shown in Example 4.11, the harmonic series diverges. It follows from Theorem 4.29 below that the alternating harmonic series converges, so it is a conditionally convergent series. Its convergence is made possible by the cancelation between terms of opposite signs.

As we show next, the convergence of an absolutely convergent series follows from the Cauchy condition. Moreover, the series of positive and negative terms in an absolutely convergent series converge separately. First, we introduce some convenient notation.

Definition 4.15. The positive and negative parts of a real number $a \in \mathbb{R}$ are given by

$$a^+ = \begin{cases} a & \text{if } a > 0, \\ 0 & \text{if } a \le 0, \end{cases} \qquad a^- = \begin{cases} 0 & \text{if } a \ge 0, \\ |a| & \text{if } a < 0. \end{cases}$$

It follows, in particular, that

$$0 \le a^+, a^- \le |a|, \qquad a = a^+ - a^-, \qquad |a| = a^+ + a^-.$$

We may then split a series of real numbers into its positive and negative parts.

Example 4.16. Consider the alternating harmonic series

$$\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

Its positive and negative parts are given by

$$\sum_{n=1}^{\infty} a_n^+ = 1 + 0 + \frac{1}{3} + 0 + \frac{1}{5} + 0 + \dots,$$
$$\sum_{n=1}^{\infty} a_n^- = 0 + \frac{1}{2} + 0 + \frac{1}{4} + 0 + \frac{1}{6} + \dots.$$

Both of these series diverge to infinity, since the harmonic series diverges and

$$\sum_{n=1}^{\infty} a_n^+ > \sum_{n=1}^{\infty} a_n^- = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}.$$

Proposition 4.17. An absolutely convergent series converges. Moreover,

$$\sum_{n=1}^{\infty} a_n$$

converges absolutely if and only if the series

$$\sum_{n=1}^{\infty} a_n^+, \qquad \sum_{n=1}^{\infty} a_n^-$$

of positive and negative terms both converge. Furthermore, in that case

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-, \qquad \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} a_n^+ + \sum_{n=1}^{\infty} a_n^-.$$

Proof. If $\sum a_n$ is absolutely convergent, then $\sum |a_n|$ is convergent, so it satisfies the Cauchy condition. Since

$$\left|\sum_{k=m+1}^{n} a_k\right| \le \sum_{k=m+1}^{n} |a_k|,$$

the series $\sum a_n$ also satisfies the Cauchy condition, and therefore it converges.

For the second part, note that

$$0 \le \sum_{k=m+1}^{n} |a_k| = \sum_{k=m+1}^{n} a_k^+ + \sum_{k=m+1}^{n} a_k^-,$$
$$0 \le \sum_{k=m+1}^{n} a_k^+ \le \sum_{k=m+1}^{n} |a_k|,$$
$$0 \le \sum_{k=m+1}^{n} a_k^- \le \sum_{k=m+1}^{n} |a_k|,$$

which shows that $\sum |a_n|$ is Cauchy if and only if both $\sum a_n^+$, $\sum a_n^-$ are Cauchy. It follows that $\sum |a_n|$ converges if and only if both $\sum a_n^+$, $\sum a_n^-$ converge. In that case, we have

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \lim_{n\to\infty} \sum_{k=1}^{n} a_k \\ &= \lim_{n\to\infty} \left(\sum_{k=1}^{n} a_k^+ - \sum_{k=1}^{n} a_k^- \right) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} a_k^+ - \lim_{n\to\infty} \sum_{k=1}^{n} a_k^- \\ &= \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-,\end{aligned}$$

and similarly for $\sum |a_n|$, which proves the proposition. □

It is worth noting that this result depends crucially on the completeness of $\mathbb{R}$.

Example 4.18. Suppose that $a_n^+, a_n^- \in \mathbb{Q}^+$ are positive rational numbers such that

$$\sum_{n=1}^{\infty} a_n^+ = \sqrt{2}, \qquad \sum_{n=1}^{\infty} a_n^- = 2 - \sqrt{2},$$

and let $a_n = a_n^+ - a_n^-$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^- = 2\sqrt{2} - 2 \notin \mathbb{Q}, \\ \sum_{n=1}^{\infty} |a_n| &= \sum_{n=1}^{\infty} a_n^+ + \sum_{n=1}^{\infty} a_n^- = 2 \in \mathbb{Q}.\end{aligned}$$

Thus, the series converges absolutely in $\mathbb{Q}$, but it doesn't converge in $\mathbb{Q}$.

4.4. The comparison test

One of the most useful ways of showing that a series is absolutely convergent is to compare it with a simpler series whose convergence is already known.

Theorem 4.19 (Comparison test)**.** Suppose that $b_n \geq 0$ and

$$\sum_{n=1}^{\infty} b_n$$

converges. If $|a_n| \leq b_n$, then

$$\sum_{n=1}^{\infty} a_n$$

converges absolutely.

Proof. Since $\sum b_n$ converges it satisfies the Cauchy condition, and since

$$\sum_{k=m+1}^{n} |a_k| \leq \sum_{k=m+1}^{n} b_k$$

the series $\sum |a_n|$ also satisfies the Cauchy condition. Therefore $\sum a_n$ converges absolutely. □

Example 4.20. The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots.$$

converges by comparison with the telescoping series in Example 4.5. We have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

and

$$0 \le \frac{1}{(n+1)^2} < \frac{1}{n(n+1)}.$$

We also get the explicit upper bound

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 2.$$

In fact, the sum is

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Mengoli (1644) posed the problem of finding this sum, which was solved by Euler (1735). The evaluation of the sum is known as the Basel problem, after Euler's birthplace in Switzerland.

It follows by comparison with $\sum 1/n^2$ that the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges for all $p > 2$. More generally, the integral test shows that the p-series converges if $p > 1$ and diverges if $0 < p \le 1$. (See Example 12.42.) This justifies the following definition.

Definition 4.21. The Riemann ζ-function is defined for $1 < s < \infty$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

For example, as stated in Example 4.20, $\zeta(2) = \pi^2/6$. In fact, there is a general formula for the value $\zeta(2n)$ of the ζ-function at even integers in terms of Bernoulli numbers, and $\zeta(2n)/\pi^{2n}$ is rational. On the other hand, the values of the ζ-function at odd numbers are harder to study. For example,

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

is called Apéry's constant, since it was proved by Apéry (1979) to be irrational, but it doesn't have a simple explicit expression like $\zeta(2)$.

The Riemann ζ-function is intimately connected with number theory and the distribution of primes. It can be extended in a unique way to an analytic (i.e., differentiable) function of a complex variable $s = \sigma + i\tau$

$$\zeta : \mathbb{C} \setminus \{1\} \to \mathbb{C},$$

where $\sigma = \Re s$ is the real part of s and $\tau = \Im s$ is the imaginary part. The ζ-function has a singularity at $s = 1$, where its modulus goes to ∞, and is equal to zero at the negative, even integers $s = -2, -4, \dots, -2n, \dots$. These zeros are called the trivial zeros of the ζ-function. Riemann (1859) made the following conjecture.

Hypothesis 4.22 (Riemann hypothesis)**.** The only zeros of the Riemann ζ-function, apart from the trivial zeros, occur on the line $\Re s = 1/2$.

Despite enormous efforts, this conjecture has neither been proved nor disproved, and it remains one of the most significant open problems in mathematics (perhaps *the* most significant open problem).

4.5. The ratio and root tests

In this section, we describe the ratio and root tests, which provide explicit sufficient conditions for the absolute convergence of a series that can be compared with a geometric series. These tests are particularly useful in studying power series, but they aren't effective in determining the convergence or divergence of series whose terms do not approach zero at a geometric rate.

Theorem 4.23 (Ratio test)**.** Suppose that (a_n) is a sequence of nonzero real numbers such that the limit

$$r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists or diverges to infinity. Then the series

$$\sum_{n=1}^{\infty} a_n$$

converges absolutely if $0 \le r < 1$ and diverges if $1 < r \le \infty$.

Proof. If $r < 1$, choose s such that $r < s < 1$. Then there exists $N \in \mathbb{N}$ such that

$$\left| \frac{a_{n+1}}{a_n} \right| < s \qquad \text{for all } n > N.$$

It follows that

$$|a_n| \le M s^n \qquad \text{for all } n > N$$

where M is a suitable constant. Therefore $\sum a_n$ converges absolutely by comparison with the convergent geometric series $\sum M s^n$.

If $r > 1$, choose s such that $r > s > 1$. There exists $N \in \mathbb{N}$ such that

$$\left| \frac{a_{n+1}}{a_n} \right| > s \qquad \text{for all } n > N,$$

so that $|a_n| \ge M s^n$ for all $n > N$ and some $M > 0$. It follows that (a_n) does not approach 0 as $n \to \infty$, so the series diverges. □

Example 4.24. Let $a \in \mathbb{R}$. Then the series

$$\sum_{n=1}^{\infty} na^n = a + 2a^2 + 3a^3 + \dots$$

converges if $|a| < 1$ and diverges if $|a| \geq 1$.

Example 4.25. Let $p > 0$ and consider the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

Then

$$\lim_{n\to\infty} \left[\frac{1/(n+1)^p}{1/n^p}\right] = \lim_{n\to\infty} \left[\frac{1}{(1+1/n)^p}\right] = 1,$$

so the ratio test is inconclusive. In this case, the series diverges if $0 < p \leq 1$ and converges if $p > 1$, which shows that either possibility may occur when the limit in the ratio test is 1.

The root test provides a criterion for convergence of a series that is closely related to the ratio test, but it doesn't require that the limit of the ratios of successive terms exists.

Theorem 4.26 (Root test)**.** Suppose that (a_n) is a sequence of real numbers and let

$$r = \limsup_{n\to\infty} |a_n|^{1/n}.$$

Then the series

$$\sum_{n=1}^{\infty} a_n$$

converges absolutely if $0 \leq r < 1$ and diverges if $1 < r \leq \infty$.

Proof. First suppose $0 \leq r < 1$. If $0 < r < 1$, choose s such that $r < s < 1$, and let

$$t = \frac{r}{s}, \qquad r < t < 1.$$

If $r = 0$, choose any $0 < t < 1$. Since $t > \limsup |a_n|^{1/n}$, Theorem 3.41 implies that there exists $N \in \mathbb{N}$ such that

$$|a_n|^{1/n} < t \qquad \text{for all } n > N.$$

Therefore $|a_n| < t^n$ for all $n > N$, where $t < 1$, so it follows that the series converges by comparison with the convergent geometric series $\sum t^n$.

Next suppose $1 < r \leq \infty$. If $1 < r < \infty$, choose s such that $1 < s < r$, and let

$$t = \frac{r}{s}, \qquad 1 < t < r.$$

If $r = \infty$, choose any $1 < t < \infty$. Since $t < \limsup |a_n|^{1/n}$, Theorem 3.41 implies that

$$|a_n|^{1/n} > t \qquad \text{for infinitely many } n \in \mathbb{N}.$$

Therefore $|a_n| > t^n$ for infinitely many $n \in \mathbb{N}$, where $t > 1$, so (a_n) does not approach zero as $n \to \infty$, and the series diverges. □

The root test may succeed where the ratio test fails, although both tests are limited to series with geometric-type convergence.

Example 4.27. Consider the geometric series with ratio 1/2,

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \dots, \qquad a_n = \frac{1}{2^n}.$$

Then (of course) both the ratio and root test imply convergence since

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{2} < 1.$$

Now consider the series obtained by switching successive odd and even terms

$$\sum_{n=1}^{\infty} b_n = \frac{1}{2^2} + \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^3} + \frac{1}{2^6} + \dots, \qquad b_n = \begin{cases} 1/2^{n+1} & \text{if } n \text{ is odd,} \\ 1/2^{n-1} & \text{if } n \text{ is even} \end{cases}$$

For this series,

$$\left| \frac{b_{n+1}}{b_n} \right| = \begin{cases} 2 & \text{if } n \text{ is odd,} \\ 1/8 & \text{if } n \text{ is even,} \end{cases}$$

and the ratio test doesn't apply. (The series still converges geometrically, however, because the the decrease in the terms by a factor of 1/8 for even n dominates the increase by a factor of 2 for odd n.) On the other hand

$$\limsup_{n\to\infty} |b_n|^{1/n} = \frac{1}{2},$$

so the ratio test still works. In fact, as we discuss in Section 4.7, since the series is absolutely convergent, every rearrangement of it converges to the same sum.

4.6. Alternating series

An alternating series is one in which successive terms have opposite signs. If the terms in an alternating series have decreasing absolute values and converge to zero, then the series converges however slowly its terms approach zero. This allows us to prove the convergence of some series which aren't absolutely convergent.

Example 4.28. The alternating harmonic series from Example 4.14 is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots.$$

The behavior of its partial sums is shown in Figure 1, which illustrates the idea of the convergence proof for alternating series.

Theorem 4.29 (Alternating series)**.** Suppose that (a_n) is a decreasing sequence of nonnegative real numbers, meaning that $0 \le a_{n+1} \le a_n$, such that $a_n \to 0$ as $n \to \infty$. Then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \dots$$

converges.

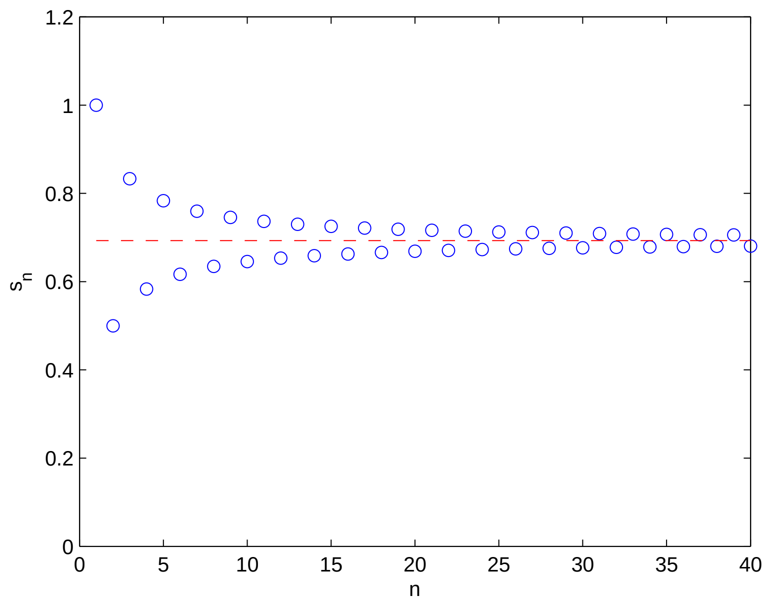

Figure 1. A plot of the first 40 partial sums S_n of the alternating harmonic series in Example 4.14. The odd partial sums decrease and the even partial sums increase to the sum of the series $\log 2 \approx 0.6931$, which is indicated by the dashed line.

Proof. Let

$$S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$$

denote the nth partial sum. If $n = 2m - 1$ is odd, then

$$S_{2m-1} = S_{2m-3} - a_{2m-2} + a_{2m-1} \leq S_{2m-3},$$

since (a_n) is decreasing, and

$$S_{2m-1} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-3} - a_{2m-2}) + a_{2m-1} \geq 0.$$

Thus, the sequence (S_{2m-1}) of odd partial sums is decreasing and bounded from below by 0, so $S_{2m-1} \downarrow S^+$ as $m \to \infty$ for some $S^+ \geq 0$.

Similarly, if $n = 2m$ is even, then

$$S_{2m} = S_{2m-2} + a_{2m-1} - a_{2m} \geq S_{2m-2},$$

and

$$S_{2m} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2m-1} - a_{2m}) \leq a_1.$$

Thus, (S_{2m}) is increasing and bounded from above by a_1, so $S_{2m} \uparrow S^- \leq a_1$ as $m \to \infty$.

Finally, note that

$$\lim_{m\to\infty} (S_{2m-1} - S_{2m}) = \lim_{m\to\infty} a_{2m} = 0,$$

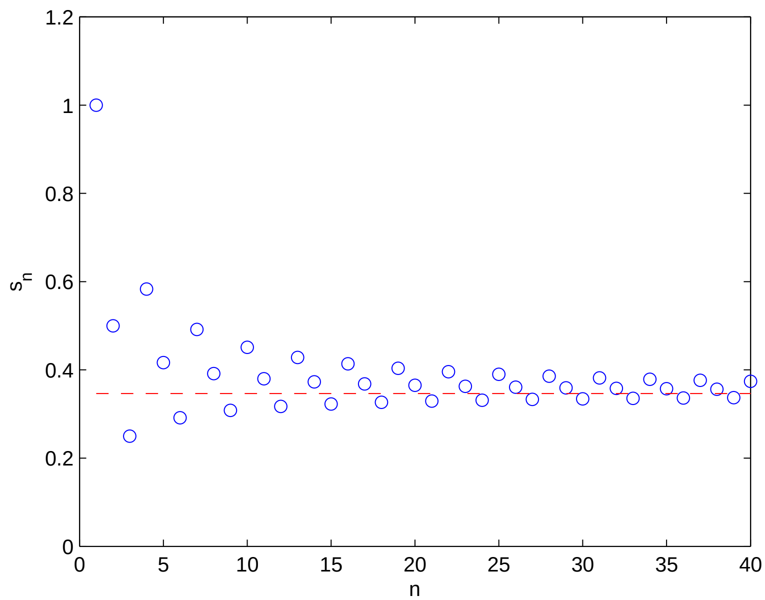

Figure 2. A plot of the first 40 partial sums S_n of the rearranged alternating harmonic series in Example 4.31. The series converges to half the sum of the alternating harmonic series, $\frac{1}{2} \log 2 \approx 0.3466$. Compare this picture with Figure 1.

so $S^+ = S^-$, which implies that the series converges to their common value. □

The proof also shows that the sum $S_{2m} \leq S \leq S_{2n-1}$ is bounded from below and above by all even and odd partial sums, respectively, and that the error $|S_n - S|$ is less than the first term a_{n+1} in the series that is neglected.

Example 4.30. The alternating p-series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$$

converges for every $p > 0$. The convergence is absolute for $p > 1$ and conditional for $0 < p \leq 1$.

4.7. Rearrangements

A rearrangement of a series is a series that consists of the same terms in a different order. The convergence of rearranged series may initially appear to be unconnected with absolute convergence, but absolutely convergent series are exactly those series whose sums remain the same under every rearrangement of their terms. On the other hand, a conditionally convergent series can be rearranged to give any sum we please, or to diverge.

Example 4.31. A rearrangement of the alternating harmonic series in Example 4.14 is

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \dots,$$

where we put two negative even terms between each of the positive odd terms. The behavior of its partial sums is shown in Figure 2. As proved in Example 12.45, this series converges to one-half of the sum of the alternating harmonic series. The sum of the alternating harmonic series can change under rearrangement because it is conditionally convergent.

Note also that both the positive and negative parts of the alternating harmonic series diverge to infinity, since

$$\begin{aligned} 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots &> \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots \\ &> \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\right), \\ \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots &= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\right), \end{aligned}$$

and the harmonic series diverges. This is what allows us to change the sum by rearranging the series.

The formal definition of a rearrangement is as follows.

Definition 4.32. A series

$$\sum_{m=1}^{\infty} b_m$$

is a rearrangement of a series

$$\sum_{n=1}^{\infty} a_n$$

if there is a one-to-one, onto function $f : \mathbb{N} \to \mathbb{N}$ such that $b_m = a_{f(m)}$.

If $\sum b_m$ is a rearrangement of $\sum a_n$ with $n = f(m)$, then $\sum a_n$ is a rearrangement of $\sum b_m$, with $m = f^{-1}(n)$.

Theorem 4.33. If a series is absolutely convergent, then every rearrangement of the series converges to the same sum.

Proof. First, suppose that

$$\sum_{n=1}^{\infty} a_n$$

is a convergent series with $a_n \geq 0$, and let

$$\sum_{m=1}^{\infty} b_m, \qquad b_m = a_{f(m)}$$

be a rearrangement.

Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that

$$0 \leq \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{N} a_k < \epsilon.$$

Since $f : \mathbb{N} \to \mathbb{N}$ is one-to-one and onto, there exists $M \in \mathbb{N}$ such that

$$\{1, 2, \ldots, N\} \subset f^{-1}(\{1, 2, \ldots, M\}),$$

meaning that all of the terms a_1, a_2,..., a_N are among the b_1, b_2,..., b_M. For example, we can take $M = \max\{m \in \mathbb{N} : 1 \leq f(m) \leq N\}$; this maximum is well-defined since there are finitely many such m (in fact, N of them).

If $m > M$, then

$$\sum_{k=1}^{N} a_k \leq \sum_{j=1}^{m} b_j \leq \sum_{k=1}^{\infty} a_k$$

since the b_j's include all the a_k's in the left sum, and all the b_j's are included among the a_k's in the right sum, and $a_k, b_j \geq 0$. It follows that

$$0 \leq \sum_{k=1}^{\infty} a_k - \sum_{j=1}^{m} b_j < \epsilon,$$

for all $m > M$, which proves that

$$\sum_{j=1}^{\infty} b_j = \sum_{k=1}^{\infty} a_k.$$

If $\sum a_n$ is a general absolutely convergent series, then from Proposition 4.17 the positive and negative parts of the series

$$\sum_{n=1}^{\infty} a_n^+, \qquad \sum_{n=1}^{\infty} a_n^-$$

converge. If $\sum b_m$ is a rearrangement of $\sum a_n$, then $\sum b_m^+$ and $\sum b_m^-$ are rearrangements of $\sum a_n^+$ and $\sum a_n^-$, respectively. It follows from what we've just proved that they converge and

$$\sum_{m=1}^{\infty} b_m^+ = \sum_{n=1}^{\infty} a_n^+, \qquad \sum_{m=1}^{\infty} b_m^- = \sum_{n=1}^{\infty} a_n^-.$$

Proposition 4.17 then implies that $\sum b_m$ is absolutely convergent and

$$\sum_{m=1}^{\infty} b_m = \sum_{m=1}^{\infty} b_m^+ - \sum_{m=1}^{\infty} b_m^- = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^- = \sum_{n=1}^{\infty} a_n,$$

which proves the result. □

Conditionally convergent series behave completely differently from absolutely convergent series under rearrangement. As Riemann observed, they can be rearranged to give any sum we want, or to diverge. Before giving the proof, we illustrate the idea with an example.

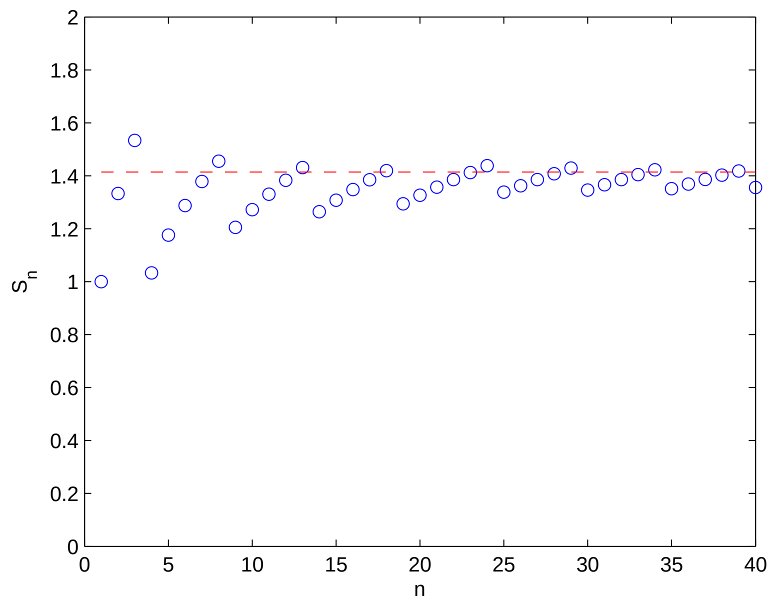

Figure 3. A plot of the first 40 partial sums S_n of the rearranged alternating harmonic series described in Example 4.34, which converges to $\sqrt{2}$.

Example 4.34. Suppose we want to rearrange the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots.$$

so that its sum is $\sqrt{2} \approx 1.4142$. We choose positive terms until we get a partial sum that is greater than $\sqrt{2}$, which gives $1 + 1/3 + 1/5$; followed by negative terms until we get a sum less than $\sqrt{2}$, which gives $1 + 1/3 + 1/5 - 1/2$; followed by positive terms until we get a sum greater than $\sqrt{2}$, which gives

$$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13};$$

followed by another negative term $-1/4$ to get a sum less than $\sqrt{2}$; and so on. The first 40 partial sums of the resulting series are shown in Figure 3.

Theorem 4.35. If a series is conditionally convergent, then it has rearrangements that converge to an arbitrary real number and rearrangements that diverge to ∞ or $-\infty$.

Proof. Suppose that $\sum a_n$ is conditionally convergent. Since the series converges, $a_n \to 0$ as $n \to \infty$. If both the positive part $\sum a_n^+$ and negative part $\sum a_n^-$ of the series converge, then the series converges absolutely; and if only one part diverges, then the series diverges (to ∞ if $\sum a_n^+$ diverges, or $-\infty$ if $\sum a_n^-$ diverges). Therefore both $\sum a_n^+$ and $\sum a_n^-$ diverge. This means that we can make sums of successive positive or negative terms in the series as large as we wish.

Suppose $S \in \mathbb{R}$. Starting from the beginning of the series, we choose successive positive or zero terms in the series until their partial sum is greater than or equal to S. Then we choose successive strictly negative terms, starting again from the beginning of the series, until the partial sum of all the terms is strictly less than S. After that, we choose successive positive or zero terms until the partial sum is greater than or equal S, followed by negative terms until the partial sum is strictly less than S, and so on. The partial sums are greater than S by at most the value of the last positive term retained, and are less than S by at most the value of the last negative term retained. Since $a_n \to 0$ as $n \to \infty$, it follows that the rearranged series converges to S.

A similar argument shows that we can rearrange a conditional convergent series to diverge to ∞ or $-\infty$, and that we can rearrange the series so that it diverges in a finite or infinite oscillatory fashion. □

The previous results indicates that conditionally convergent series behave in many ways more like divergent series than absolutely convergent series.

4.8. The Cauchy product

In this section, we prove a result about the product of absolutely convergent series that is useful in multiplying power series.

Definition 4.36. The Cauchy product of the series

$$\sum_{n=0}^{\infty} a_n, \qquad \sum_{n=0}^{\infty} b_n$$

is the series

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right).$$

Here, it is convenient to begin numbering the terms of the series at $n = 0$. The Cauchy product arises formally by term-by-term multiplication and rearrangement:

$$\begin{aligned}
&(a_0 + a_1 + a_2 + a_3 + \dots)(b_0 + b_1 + b_2 + b_3 + \dots) \\
&\quad = a_0 b_0 + a_0 b_1 + a_0 b_2 + a_0 b_3 + \dots + a_1 b_0 + a_1 b_1 + a_1 b_2 + \dots \\
&\qquad\quad + a_2 b_0 + a_2 b_1 + \dots + a_3 b_0 + \dots \\
&\quad = a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) \\
&\qquad\quad + (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0) + \dots .
\end{aligned}$$

In general, writing $m = n - k$, we have formally that

$$\left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} a_k b_m = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k}.$$

There are no convergence issues about the individual terms in the Cauchy product, since $\sum_{k=0}^{n} a_k b_{n-k}$ is a finite sum.

Theorem 4.37. If the series

$$\sum_{n=0}^{\infty} a_n, \qquad \sum_{n=0}^{\infty} b_n$$

are absolutely convergent, then the Cauchy product is absolutely convergent and

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) = \left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right).$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{N} \left| \sum_{k=0}^{n} a_k b_{n-k} \right| &\leq \sum_{n=0}^{N} \left(\sum_{k=0}^{n} |a_k||b_{n-k}| \right) \\
&\leq \sum_{n=0}^{N} \left(\sum_{k=0}^{N} |a_k||b_{n-k}| \right) \\
&\leq \left(\sum_{k=0}^{N} |a_k| \right) \left(\sum_{m=0}^{N} |b_m| \right) \\
&\leq \left(\sum_{n=0}^{\infty} |a_n| \right) \left(\sum_{n=0}^{\infty} |b_n| \right).
\end{aligned}$$

Thus, the Cauchy product is absolutely convergent, since the partial sums of its absolute values are bounded from above.

Since the series for the Cauchy product is absolutely convergent, any rearrangement of it converges to the same sum. In particular, the subsequence of partial sums given by

$$\left(\sum_{n=0}^{N} a_n \right) \left(\sum_{n=0}^{N} b_n \right) = \sum_{n=0}^{N} \sum_{m=0}^{N} a_n b_m$$

corresponds to a rearrangement of the Cauchy product, so

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) = \lim_{N\to\infty} \left(\sum_{n=0}^{N} a_n \right) \left(\sum_{n=0}^{N} b_n \right) = \left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right).$$

□

In fact, since the series of term-by-term products of absolutely convergent series converges absolutely, every rearrangement of the product series — not just the one in the Cauchy product — converges to the product of the sums,.

4.9. The irrationality of e

In this section, we use series to prove that e is an irrational number. In Proposition 3.32, we defined $e \approx 2.71828\ldots$ as the limit

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^n.$$

We first prove an alternative expression for e as the sum of a series.

Proposition 4.38.

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Proof. Using the binomial theorem, as in the proof of Proposition 3.32, we find that

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= 2 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \\ &\qquad + \dots + \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{k-1}{n}\right) \\ &\qquad + \dots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\frac{2}{n}\cdot\frac{1}{n} \\ &< 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{n!}.\end{aligned}$$

Taking the limit of this inequality as $n \to \infty$, we get that

$$e \le \sum_{n=0}^{\infty} \frac{1}{n!}.$$

To get the reverse inequality, we observe that for every $k \le n$,

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &\ge 2 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \\ &\qquad + \dots + \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{k-1}{n}\right).\end{aligned}$$

Fixing k and taking the limit as $n \to \infty$, we get that

$$e \ge \sum_{j=0}^{k} \frac{1}{j!}.$$

Then, taking the limit as $k \to \infty$, we find that

$$e \ge \sum_{n=0}^{\infty} \frac{1}{n!},$$

which proves the result. □

This series for e is very rapidly convergent. The next proposition gives an explicit error estimate.

Proposition 4.39. For every $n \in \mathbb{N}$,

$$0 < e - \sum_{k=0}^{n} \frac{1}{k!} < \frac{1}{n \cdot n!}.$$

Proof. The lower bound is obvious. For the upper bound, we estimate the tail of the series by a geometric series:

$$\begin{aligned}
e - \sum_{k=0}^{n} \frac{1}{k!} &= \sum_{k=n+1}^{\infty} \frac{1}{k!} \\
&= \frac{1}{n!}\left(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \dots + \frac{1}{(n+1)(n+2)\dots(n+k)} + \dots\right) \\
&< \frac{1}{n!}\left(\frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots + \frac{1}{(n+1)^k} + \dots\right) \\
&< \frac{1}{n!}\cdot\frac{1}{n}.
\end{aligned}$$

□

Theorem 4.40. The number e is irrational.

Proof. Suppose for contradiction that $e = p/q$ for some $p, q \in \mathbb{N}$. From Proposition 4.39,

$$0 < \frac{p}{q} - \sum_{k=0}^{n} \frac{1}{k!} < \frac{1}{n \cdot n!}$$

for every $n \in \mathbb{N}$. Multiplying this inequality by $n!$, we get

$$0 < \frac{p \cdot n!}{q} - \sum_{k=0}^{n} \frac{n!}{k!} < \frac{1}{n}.$$

The middle term is an integer if $n \geq q$, which is impossible since it lies strictly between 0 and $1/n$. □

A real number is said to be algebraic if it is a root of a polynomial with integer coefficients, otherwise it is transcendental. For example, every rational number $r = p/q$, where $p, q \in \mathbb{Z}$ and $q \neq 0$, is algebraic, since it is the solution of $qx - p = 0$; and $\sqrt{2}$ is an irrational algebraic number, since it is a solution of $x^2 - 2 = 0$. It's possible to prove that e is not only irrational but transcendental, but the proof is harder. Two other explicit examples of transcendental numbers are π and $2^{\sqrt{2}}$.

There is a long history of the study of irrational and transcendental numbers. Euler (1737) proved the irrationality of e, and Lambert (1761) proved the irrationality of π. The first proof of the existence of a transcendental number was given by Liouville (1844), who showed that

$$\sum_{n=1}^{\infty} \frac{1}{10^{n!}} = 0.110001000000000000000001000\dots$$

is transcendental. The transcendence of e was proved by Hermite (1873), the transcendence of π by Lindemann (1882), and the transcendence of $2^{\sqrt{2}}$ independently by Gelfond and Schneider (1934).

Cantor (1878) observed that the set of algebraic numbers is countably infinite, since there are countable many polynomials with integer coefficients, each such polynomial has finitely many roots, and the countable union of countable sets is countable. This proved the existence of uncountably many transcendental numbers without exhibiting any explicit examples, which was a remarkable result given the difficulties mathematicians had encountered (and still encounter) in proving that specific numbers were transcendental.

There remain many unsolved problems in this area. For example, it's not known if the Euler constant

$$\gamma = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right)$$

from Example 4.11 is rational or irrational.