Improper Integrals and the Fundamental Theorem of Calculus

One of the applications of integration is to determine probabilities that involve a continuous random variable, X:

Let the continuous random variable X have a probability density function, pdf, *f(x)* defined over the interval [*a, b*]. Then to find the probability that X lies in the interval $x_1 \le X \le x_2$ where $a \le x_1 \le X \le x_2 \le b$ we find the area of the region enclosed by the curve $y = f(x)$, the x-axis and the lines $x = x_1$ and $x = x_2$.

EXAMPLE: The continuous random variable X has a probability density function defined by $f(x) = \begin{cases} k(2-x)^2, 0 \le x \le 2 \\ 0, x < 0, x > 2 \end{cases}$

a. Find *k*

As

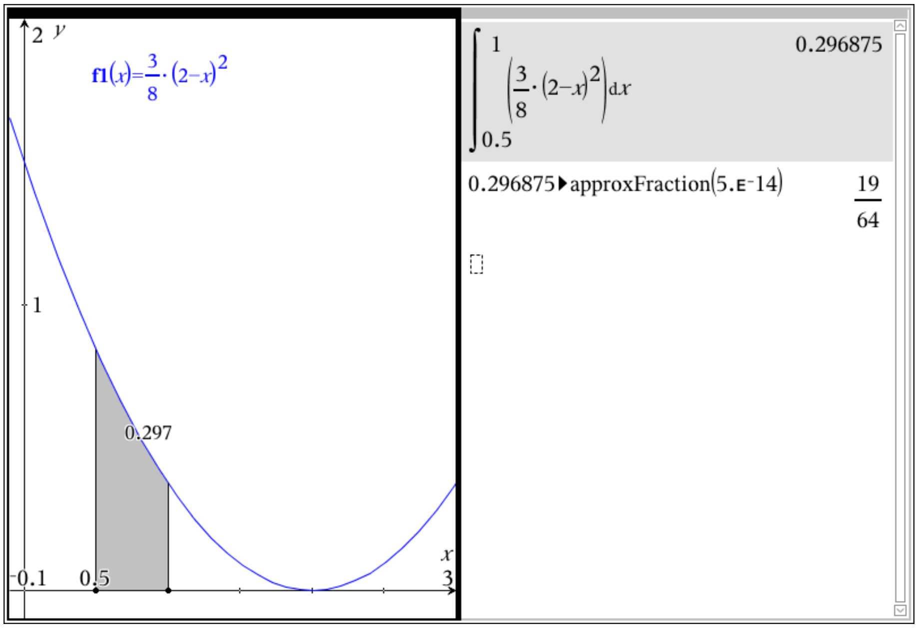
$$f(x) = \int_0^2 k(2-x)^2\,dx = 1$$

$$\Leftrightarrow k\left[-\frac{1}{3}(2-x)^3\right]_0^2 = 1$$

$$\Leftrightarrow k\left(0 + \frac{8}{3}\right) = 1$$

$$k = \frac{3}{8}$$

b. Find the probability that $0.5 \le X \le 1$



c. Find the probability that $x > 1 | X > 0.5$ (Probability that X > 1 given that X > 0.5.

$$P\left(X > 1 | X > 0.5\right) = \frac{P(x > 1) \cap P(x > 0.5)}{P(x > 0.5)} = \frac{P(x > 1)}{P(x > 0.5)}$$

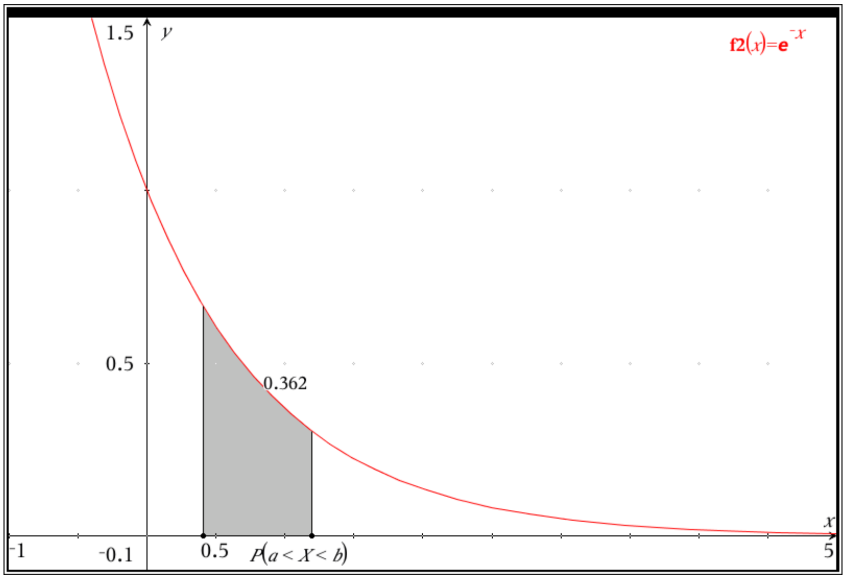
$$P(X > 1) = \int_1^2 \frac{3}{8}(2-x)^2\, dx$$

$$P(X > 1) = \left[-\frac{1}{8}(2-x)^3\right]_1^2 = \left[0 + \frac{1}{8}\right]$$

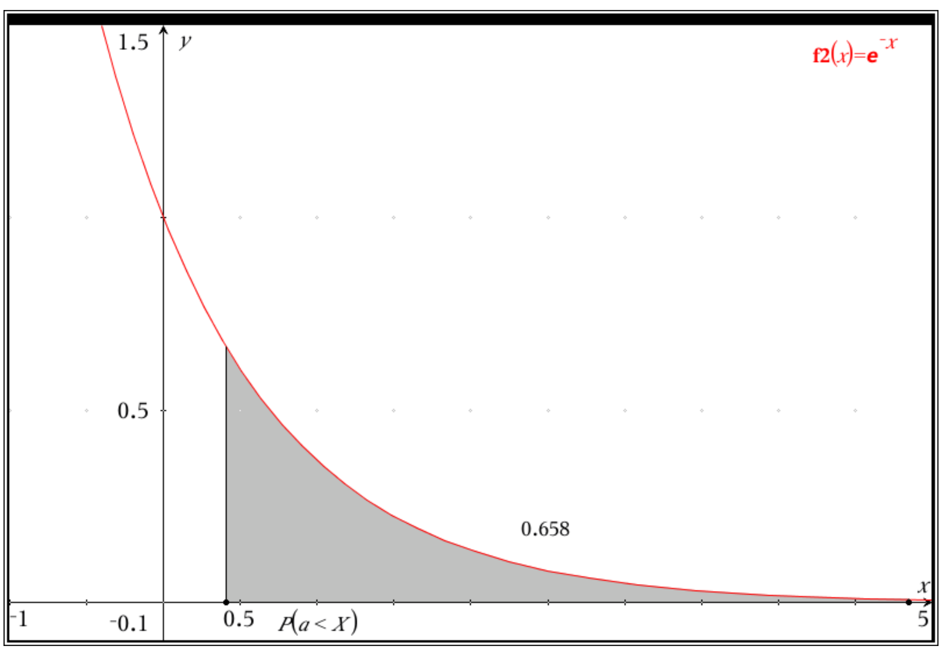
$$P(X > 0.5) = \int_{0.5}^2 \frac{3}{8}(2-x)^2\, dx$$

$$P(X > 0.5) = \left[-\frac{1}{8}(2-x)^3\right]_{0.5}^2 = \left[0 + \frac{27}{64}\right]$$

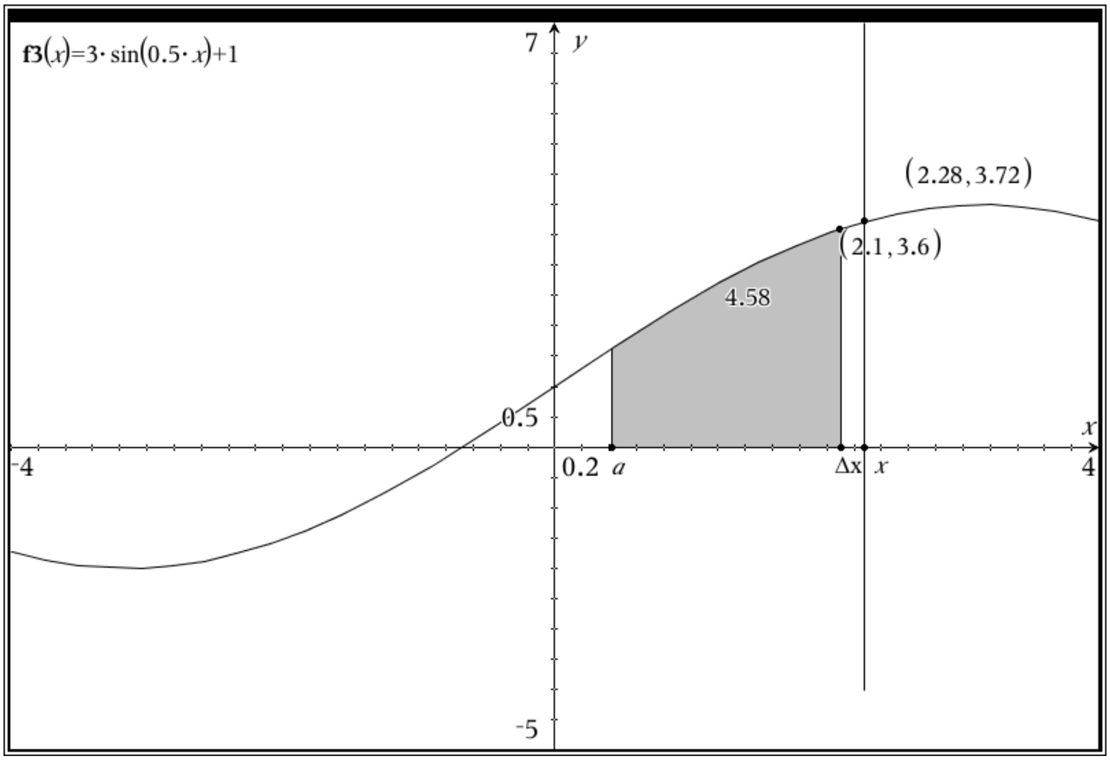
$$\frac{P(x > 1)}{P(x > 0.5)} = \frac{1/8}{27/64} = \frac{8}{27}$$



But, what if we want $P(x > a)$ where the probability density function is non-zero for all real numbers?



You have seen that the area under a continuous curve, $f(x)$ between the limits $x = a$ and $x = b$ is found by calculating the definite integral $A = \int_a^b f(x)dx$. If we allow the upper limit to vary, then $A(x) = \int_a^x f(x)dx$. Because we use *x* as a limit of integration, we cannot use it as the variable we are integrating with respect to – we simply use another variable referred to as a dummy variable.



If the change in *x* is very small, the new area is approximated by a rectangle with area $\Delta A \approx f(x)\Delta x \Leftrightarrow \frac{\Delta A}{\Delta x} \approx f(x)$. In the limit, $f(x) = \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = \frac{dA}{dx}$ or

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

This shows that $f(x)$ is the derivative of the area function, $A(x)$ with respect to *x*. But the function $A(x)$ is not the only function whose derivative is $f(x)$. For any function, $g(x)$ such that $g'(x) = f(x)$, we must have $A(x) = g(x) + c$. This can be proved using the Mean Value Theorem.

$$A(a) = \int_a^a f(t)\,dt = 0$$
$$g(a) + c = 0$$
$$\Rightarrow c = -g(a)$$
$$\therefore A(x) = g(x) - g(a)$$

Letting $x = b$

$$A(b) = g(b) - g(a)$$

$$A(b) = \int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = g(b) - g(a)$$

Fundamental Theorem of Calculus

For a continuous function $f(x)$ on the interval $[a,b]$:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

For any function $g(x)$ such that $g'(x) = f(x)$

$$\int_a^b f(x)\,dx = g(b) - g(a)$$

EXAMPLE: Find $\frac{d}{dx}\int_a^x \cos(t^2)\,dt$

While we cannot integrate $\cos(t^2)$, using the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_a^x \cos(t^2)\,dt = \cos(x^2)$$

EXAMPLE: Find $\frac{d}{da}\int_a^b f(y)\,dy$

$$\begin{aligned} \frac{d}{da}\int_a^b f(y)\,dy &= \frac{d}{da}\left(-\int_b^a f(y)\,dy\right) \\ &= -\frac{d}{da}\left(\int_b^a f(y)\,dy\right) \\ &= -f(a) \end{aligned}$$

EXAMPLE: Find $\frac{d}{dx}\int_a^b f(z)dz$. Let $g'(z) = f(z)$. Then

$$\int_a^b f(z)dz = g(b) - g(a)$$

and

$$\frac{d}{dx}\int_a^b f(z)dz = \frac{d}{dx}\left(g(b) - g(a)\right) = 0\,.$$

We can also apply the FTC to a function with two variables, $f(x,y)$ by integrating with respect to just one of these variables, treating the other as a constant.

EXAMPLE: Find $\frac{d}{dx}\int_a^b x^2\cos(y)dy$

$$\int_a^b x^2\cos(y)dy = x^2\int_a^b \cos(y)dy$$

$$\int_a^b x^2\cos(y)dy = x^2\left[\sin y\right]_a^b$$

$$= x^2\left(\sin(b) - \sin(a)\right)$$

$$\frac{d}{dx}\int_a^b x^2\cos(y)dy = \frac{d}{dx}\left(x^2\left(\sin(b) - \sin(a)\right)\right)$$

$$= 2x\left(\sin(b) - \sin(a)\right)$$

EXAMPLE: Find $\int_a^b \frac{d}{dx}x^2\cos(y)dy$

$$\int_a^b \frac{d}{dx}x^2\cos(y)dy = \int_a^b 2x\cos(y)dy$$

$$= 2x\int_a^b \cos(y)dy$$

$$= 2x\left[\sin x\right]_a^b$$

$$= 2x(\sin b - \sin a)$$