Indeterminate Forms and L'Hôpital's Rule

The forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ are *indeterminate* because they do not guarantee that a limit exists, nor do they indicate what the limit is if one does exist.

| *Indeterminate form* | *Limit* | *Algebraic Technique* |
|---|---|---|
| $\frac{0}{0}$ | $\lim_{x\to -1} \frac{2x^2-2}{x+1} = \lim_{x\to -1} 2(x-1)$ | Divide numerator and denominator by x + 1 |
| $\frac{\infty}{\infty}$ | $\lim_{x\to \infty} \frac{3x^2-1}{2x^2+1} = \lim_{x\to \infty} \frac{3-\left(1/x^2\right)}{2+\left(1/x^2\right)}$ | Divide numerator and denominator by highest degree of x |
| $\frac{0}{0}$ | $\lim_{x\to c^+} \frac{f(x)}{g(x)} = \lim_{x\to c^+} \frac{f'(x)}{g'(x)}$<br>$g'(x)\neq 0$<br>$\lim_{x\to \infty} \frac{\ln x}{x} = \lim_{x\to \infty} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x} = \lim_{x\to \infty} \frac{1/x}{1} = 0$<br>$0/0$ | Factor numerator and divide |

Not all indeterminate forms can be evaluated by algebraic manipulation. This is particularly true when *bot*h transcendental and algebraic functions are involved.

For instance, $\lim_{x\to 0} \frac{e^{2x}-1}{x}$ produces the indeterminate form $\frac{0}{0}$.

**THEOREM L'Hôpital's Rule**

Let *f* and *g* be functions that are differentiable on an open interval (*a,b*) containing *c*, except possibly at *c* itself. Assume $g'(x)\neq 0$ for all *x* in (*a,b*), except possibly at *c* itself.

If the limit of *f(x)/g(x)* produces the indeterminate form $\frac{0}{0}$, then

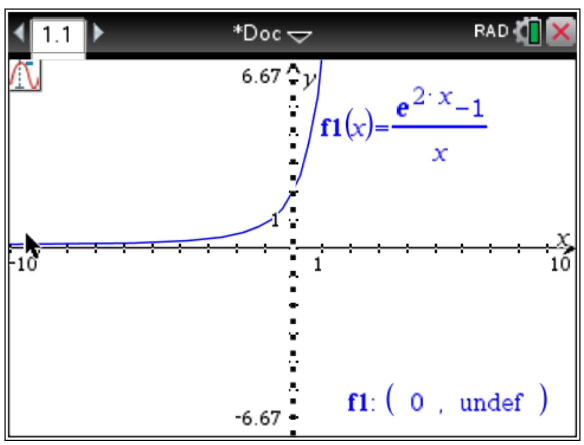
$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$$

provided that the limit on the right exists (or is infinite). This result is also true if the limit of *f(x)/g(x)* produces any one of the indeterminate forms $\infty/\infty$, $(-\infty)/\infty$, $\infty/(-\infty)$ or $(-\infty)/(-\infty)$. L'Hôpital's Rule can also be applied to one-sided limits. If the the limit as *x* approaches *c* from the right produces the indeterminate form $0/0$ then

$$\lim_{x\to c^+}\frac{f(x)}{g(x)}=\lim_{x\to c^+}\frac{f'(x)}{g'(x)}.$$

<u>EXAMPLE Indeterminate form $0/0$</u>

Evaluate $\lim_{x\to 0}\frac{e^{2x}-1}{x}$




$$\lim_{x\to 0}\frac{e^{2x}-1}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}e^{2x}-1}{\frac{d}{dx}x}$$

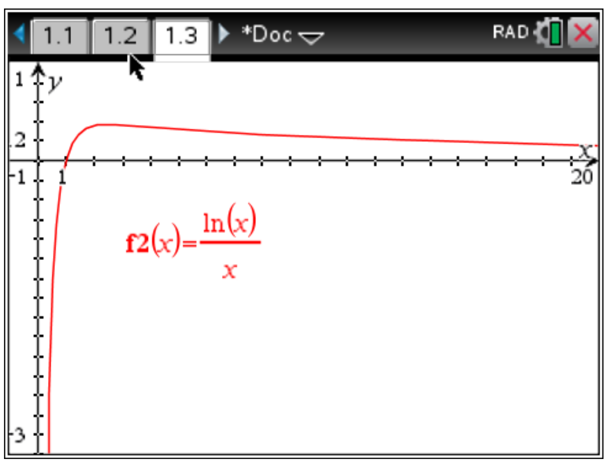
$$=\lim_{x\to 0}\frac{2e^{2x}}{1}=2$$

Another form of L'Hôpital's Rule states that if the limit of *f(x)/g(x)* produces the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ as $x \to \infty$, then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

<u>EXAMPLE Indeterminate form $\infty/\infty$</u>

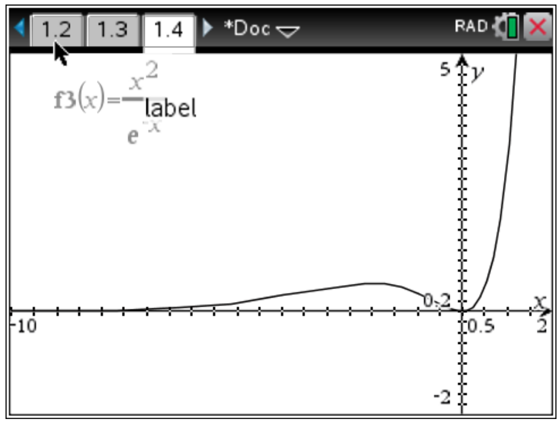
Evaluate $\lim_{x\to\infty}\frac{\ln x}{x}$




$$\lim_{x\to-\infty}\frac{x^2}{e^{-x}} = \lim_{x\to\infty}\frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x} = \lim_{x\to\infty}\frac{1/x}{1} = 0$$

EXAMPLE Applying L'Hôpital's Rule more than once

Evaluate $\lim\limits_{x\to-\infty}\frac{x^2}{e^{-x}}$




$$\lim_{x\to-\infty}\frac{x^2}{e^{-x}}=\lim_{x\to-\infty}\frac{\frac{d}{dx}x^2}{\frac{d}{dx}e^{-x}}=\lim_{x\to-\infty}\frac{2x}{-e^{-x}}=\infty/\infty$$

$$\lim_{x\to\infty}\frac{2x}{-e^{-x}}=\lim_{x\to-\infty}\frac{\frac{d}{dx}2x}{\frac{d}{dx}-e^{-x}}=\lim_{x\to-\infty}\frac{2x}{e^{-x}}=0$$

There are other indeterminate forms such as

$$\lim_{x\to1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to1^+}\left(\frac{(x-1)-\ln x}{\ln x(x-1)}\right)$$

$$=\lim_{x\to1^+}\left(\frac{\frac{d}{dx}(x-1)-\ln x}{\frac{d}{dx}\ln x(x-1)}\right)=\lim_{x\to1^+}\left(\frac{1-1/x}{1/x\,(x-1)+\ln x}\right)$$

$$\lim_{x\to1^+}\left(\frac{x-1}{x-1+x\ln x}\right)=\lim_{x\to1^+}\left(\frac{\frac{d}{dx}(x-1)}{\frac{d}{dx}(x-1+x\ln x)}\right).$$

$$=\lim\left(\frac{1}{\ldots}\right)=\frac{1}{2}$$

EXAMPLE Indeterminate form $0 \cdot \infty$.

Evaluate

$$\lim_{x \to \infty} e^{-x}\sqrt{x} = \lim_{x \to \infty} \frac{\sqrt{x}}{e^x}$$

$$\lim_{x \to \infty} \frac{\sqrt{x}}{e^x} = \lim_{x \to \infty} \frac{1 \big/ 2\sqrt{x}}{e^x} = \lim_{x \to \infty} \frac{1}{2\sqrt{x}e^x} = 0$$

Because direct substitution produces the indeterminate form $0 \cdot \infty$, rewrite the function to fit the form $0/0$ or $\infty/\infty$.

$$\lim_{x \to \infty} \frac{\sqrt{x}}{e^x} = \lim_{x \to \infty} \frac{1 \big/ 2\sqrt{x}}{e^x} = \lim_{x \to \infty} \frac{1}{2\sqrt{x}e^x} = 0$$

EXAMPLE Indeterminate form $1^{\infty}$

Evaluate $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

$$\ln y = \ln\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]$$

$$= \lim_{x\to\infty}\left[x\ln\left(1+\frac{1}{x}\right)\right]$$

$$= \lim_{x\to\infty}\left[\frac{\ln\left(1+ {}^{1}\!/_{x}\right)}{{}^{1}\!/_{x}}\right]$$

$$= \lim_{x\to\infty}\left[\frac{-{}^{1}\!/_{x^2}\left[1/\left(1+{}^{1}\!/_{x}\right)\right]}{-{}^{1}\!/_{x^2}}\right]$$

$$= \frac{1}{1}$$

$$\ln y = 1 \therefore y = e$$

EXAMPLE Indeterminate form $0^0$

Evaluate $\lim_{x\to 0^+}\left(\sin x\right)^{x}$

$$y=\lim_{x\to 0^+}(\sin x)^x$$

$$\ln y=\ln\left[\lim_{x\to 0^+}(\sin x)^x\right]$$

$$=\lim_{x\to 0^+}\left[\ln(\sin x)^x\right]$$

$$=\lim_{x\to 0^+}\left[x\ln(\sin x)\right]$$

$$=\lim_{x\to 0^+}\left[\frac{\ln(\sin x)}{1/x}\right]$$

$$=\lim_{x\to 0^+}\left[\frac{\cot x}{-1/x^2}\right]$$

$$=\lim_{x\to 0^+}\left[\frac{-x^2}{\tan x}\right]$$

$$=\lim_{x\to 0^+}\left[\frac{-2x}{\sec^2 x}\right]=0$$

$$\ln y=0, \therefore e^0=1$$

EXAMPLE Indeterminate form $\infty - \infty$

Evaluate $\lim_{x \to 1^+} \left( \frac{1}{\ln x} - \frac{1}{x-1} \right)$

$$\lim_{x \to 1^+} \left( \frac{1}{\ln x} - \frac{1}{x-1} \right) = \lim_{x \to 1^+} \left( \frac{(x-1) - \ln x}{\ln x (x-1)} \right)$$

$$= \lim_{x \to 1^+} \left( \frac{\frac{d}{dx}(x-1) - \ln x}{\frac{d}{dx} \ln x (x-1)} \right) = \lim_{x \to 1^+} \left( \frac{1 - \frac{1}{x}}{\frac{1}{x}(x-1) + \ln x} \right)$$

$$\lim_{x \to 1^+} \left( \frac{x-1}{x-1+x\ln x} \right) = \lim_{x \to 1^+} \left( \frac{\frac{d}{dx}(x-1)}{\frac{d}{dx}(x-1+x\ln x)} \right)$$

$$= \lim_{x \to 1^+} \left( \frac{1}{1 + \ln x + \left( \frac{1}{x} \right)} \right) = \frac{1}{2}$$