Infinite Series

An infinite series is an expression of the form $a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k$

The partial sums form a sequence

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$.$$
$$.$$
$$.$$
$$s_n = \sum_{k=1}^{n} a_k$$

of real numbers, each defined as a finite sum. If the sequence of partial sums has a limit as $n \to \infty$, the series converges to the sum *S* and we write

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k = S$$

Otherwise, the series diverges.

EXAMPLES:

$\sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^n$ is a geometric series with $a = 2$ and $r = \frac{2}{3}$. The n[th] partial sum is given by $S = \frac{a(1-r^n)}{1-r}$. $S = \frac{2\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\frac{2}{3}} = 6\left(1-\left(\frac{2}{3}\right)^n\right)$ Now take the limit as $n \to \infty$. $\left(\frac{2}{3}\right)^n \to 0$ as $n \to \infty$ and therefore $S_n \to 6(1-0) = 6$. Since the series of partial sums converges, so does $\sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^n$

$\sum_{k-1}^{\infty}\frac{3-k}{4}=\frac{1}{2}+\frac{1}{4}+0-\frac{1}{4}-\frac{1}{2}\cdots$ This is an arithmetic series with $a=\frac{1}{2}$ and $d=-\frac{1}{4}$. So the n[th] partial sum is given by: $S_n=\frac{n}{2}\left(2a_1+(n-1)d\right)$

$$=\frac{n}{2}\left(2\cdot\frac{1}{2}+(n-1)\left(-\frac{1}{4}\right)\right)$$

$$=\frac{n}{2}\left(1-\left(\frac{(n-1)}{4}\right)\right)$$

$$=\frac{n}{2}-\frac{n^2}{8}+\frac{n}{2}$$

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$$=\frac{5n-n^2}{8}$$

$$\text{Letting } n\to\infty, S_n=\frac{5n-n^2}{8}\to-\infty$$

Since the partial sum diverges, so does $\sum_{k-1}^{\infty}\frac{3-k}{4}$.

Whenever an infinite series, $\sum_{k=1}^{\infty}a_k$, converges, the n[th] term as $n\to\infty$ has a limit of zero.

If the **infinite series**, $\sum_{k=1}^{\infty}a_k=a_1+a_2+a_3+\cdots+a_n+\cdots$ converges, then $\lim_{k\to\infty}a_k=0$. If $\lim_{k\to\infty}a_k\neq 0$, the series must diverge. To determine whether a series converges, the first step is to determine whether it meets the *necessary* but not *sufficient* condition $\lim_{k\to\infty} a_k = 0$.

The n^th term test for divergence: If the sequence $\{a_n\}$ does not converge to zero, then the infinite series $\sum_{n=1}^{\infty} a_n$ diverges.

Below is an example of insufficiency of the condition $\lim_{k\to\infty} a_k = 0$

The Harmonic Series, $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$, would appear to converge but, in fact diverges. This can be proven algebraically by contradiction.

Assume $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots = S$, a finite sum. Then

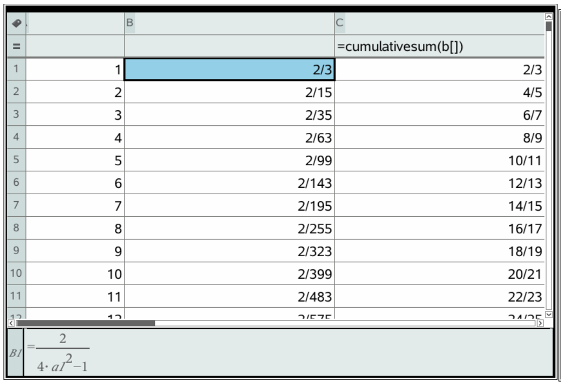
$$
\begin{aligned}
S &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots \\
&= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) + \left(\frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{n} + \frac{1}{n+1}\right) + \cdots \\
&> \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{8} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2n} + \frac{1}{2n}\right) + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots = S.
\end{aligned}
$$

$$\left(\frac{1}{2} + \frac{1}{2}\right) = 1, \ \left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{2}, \ \left(\frac{1}{6} + \frac{1}{6}\right) = \frac{1}{3} \cdots$$

This would mean that $S > S$, a contradiction. The assumption that the harmonic series has a finite sum is false.

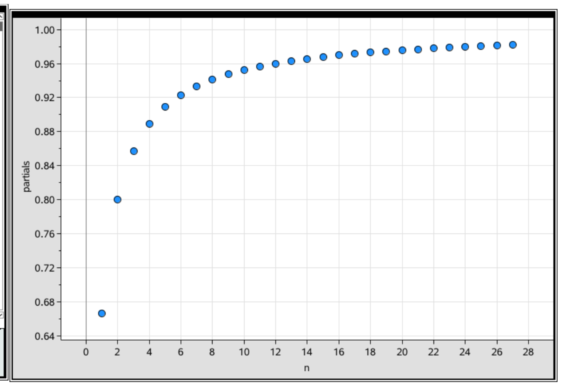
In the geometric series $\sum_{n=1}^{\infty} a_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots$ convergence is dependent upon the value of the common ratio. The series has a finite sum, $S = \frac{a}{1-r}$ if $|r| < 1$, and diverges if $|r| \geq 1$. The interval $-1 < r < 1$ is called the ***interval of convergence.***

EXAMPLE: Determine whether $\sum_{n=1}^{\infty} \frac{2}{4n^2-1}$ converges.



	B	C
=		=cumulativesum(b[])
1	2/3	2/3
2	2/15	4/5
3	2/35	6/7
4	2/63	8/9
5	2/99	10/11
6	2/143	12/13
7	2/195	14/15
8	2/255	16/17
9	2/323	18/19
10	2/399	20/21
11	2/483	22/23

$B1 = \frac{2}{4 \cdot a1^2 - 1}$



It appears that the series is converging to 1 but can we be sure using only graphical evidence?

Using partial fraction decomposition we can show that $\sum_{n=1}^{\infty} \frac{2}{4n^2-1} = \frac{1}{2n-1} - \frac{1}{2n+1}$.

$$S_n = \left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{9}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)+\cdots$$

$$=1-\left(\frac{1}{3}-\frac{1}{3}\right)-\left(\frac{1}{5}-\frac{1}{5}\right)-\left(\frac{1}{7}-\frac{1}{7}\right)-\cdots-\left(\frac{1}{2n-1}-\frac{1}{2n-1}\right)-\frac{1}{2n+1}$$

$$=1-\frac{1}{2n+1}$$

$$\lim_{n\to\infty}\left(1-\frac{1}{2n+1}\right)=1$$

The original series is called a **telescoping** series where the only remaining terms are the first and last after the sums or differences are calculated.