Infinite Series
Some Tests for Divergence and Convergence

Divergence Test: If $\lim_{n\to\infty} u_n \neq 0$ or if the limit does not exist, the series $\sum_{n=1}^{\infty} u_n$ is divergent.

EXAMPLE: Show that the series $\sum_{n=1}^{\infty} \frac{n^2+3n+1}{4n^2+3}$ diverges.

$$u_n = \frac{n^2+3n+1}{4n^2+3}$$

$$= \frac{1+\frac{3}{n}+\frac{1}{n^2}}{4+\frac{3}{n^2}}$$

$$\lim_{n\to\infty} \frac{1+\frac{3}{n}+\frac{1}{n^2}}{4+\frac{3}{n^2}} = \frac{1}{4} \neq 0$$

Comparison Test: Given two series of positive terms $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ such that $a_k \leq b_k$ for all *k* belonging to the set of positive integers, then if:

- $\sum_{k=1}^{\infty} b_k$ **is convergent to a limit *S*, $\sum_{k=1}^{\infty} a_k$ is also convergent to a limit *T*, where** $T \leq S$
- $\sum_{k=1}^{\infty} a_k$ **is divergent, so is** $\sum_{k=1}^{\infty} b_k$

For example, $1+\frac{1}{4}+\frac{1}{10}+\frac{1}{28}+\frac{1}{82}+\frac{1}{244}+\cdots$ is similar to the geometric series $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+\cdots$ with a common ratio of $\frac{1}{3}$ and an infinite sum of

$$S_\infty = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}.$$

Each term in the first sequence is less than or equal to each term of the second so by the comparison test, the sequence converges to a sum less than $\frac{3}{2}$.

The Integral as the limit of sums

Recall that the area under a curve can be approximated using Riemann Sums and an exact value for the area can be found by taking the limit as the number of rectangles used approaches infinity, or as the change in *x* approaches zero.

$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

where $f(a+i\Delta x)$ gives the height of a rectangle at its right endpoint and $\Delta x_i = \frac{b-a}{n}$ is the width of each rectangle in the interval $[a,b]$. The definite integral $\int_a^b f(x)\,dx = F(b)-F(a) = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x_i$

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EXAMPLE: For the function $f(x)=e^x$ and the partition of the interval $[0,1]$ by $c_i = \frac{i-1}{n}, i = 1,2,3,\ldots n$ use a suitable Riemann sum to show that

$$\lim_{n\to\infty}\frac{1}{n}\left(1+e^{1/n}+e^{2/n}+e^{3/n}+\cdots e^{n-1/n}\right)=e-1$$

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n} e^{c_i}\frac{1}{n} = \sum_{i=1}^{n} e^{\frac{i-1}{n}}\frac{1}{n} = \frac{1}{n}\left(1+e^{1/n}+e^{2/n}+e^{3/n}+\cdots e^{n-1/n}\right)$$

Hence, $\lim_{n\to\infty}\frac{1}{n}\left(1+e^{1/n}+e^{2/n}+e^{3/n}+\cdots e^{n-1/n}\right) = \int_0^1 e^x\,dx = e-1$

Improper Integrals

The definite integral $\int_a^b f(x)dx$ requires that for the interval $[a,b]$, *a* and *b* must be real numbers and that *f* must be continuous on the interval. Integrals that do not meet these requirements are called *improper integrals.*

Definition: An improper integral $\int_a^b f(x)dx$ has infinity as one or both of its endpoints, or contains a finite number of infinite discontinuities in the interval $[a,b]$.

Consider the function $y = e^{-x/2}$.

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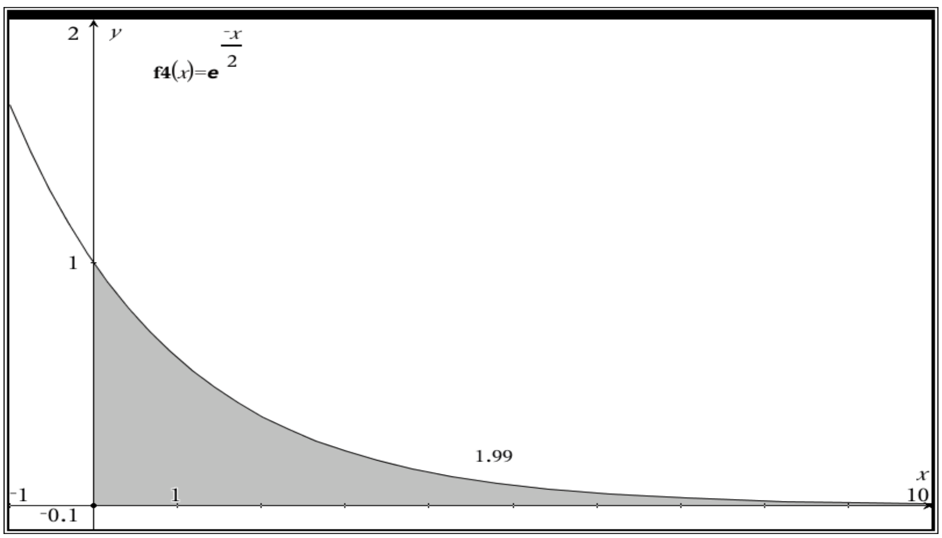


$$A(b) = \int_0^b e^{-x/2}\,dx = \left[-2e^{-x/2}\right]_0^b = -2e^{-b/2} + 2$$

$$\lim_{b\to\infty} A(b) = \lim_{b\to\infty} -2e^{-b/2} + 2$$

Hence, the area under the curve from $a = 0$ to $b = \infty$ is

$$\int_0^\infty e^{-x/2}\,dx = \lim_{b\to\infty}\int_0^b e^{-x/2}\,dx = 2$$

More generally, if *f(x)* is continuous on the interval $[a,\infty)$ then

$$\int_0^\infty f(x)dx = \lim_{b\to\infty}\int_0^b f(x)dx$$

Convergence of an improper integral

If the limit exists, the improper integral converges to the limiting value. If the limit fails to exist, the improper integral diverges.

EXAMPLES: Evaluate, if possible, the following improper integrals by discussing convergence.

a) $\int_1^{\infty} \frac{1}{x} dx$ b) $\int_0^{\infty} \frac{1}{1+x^2} dx$

a) Diverges because

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x} dx$$

$$= \lim_{b\to\infty} \ln[x]_1^b = \lim_{b\to\infty} (\ln b) = \infty$$

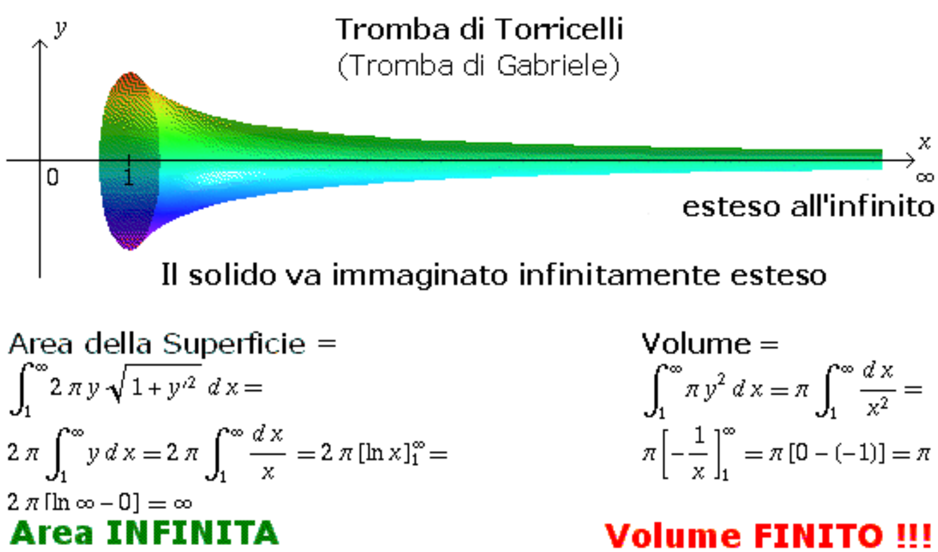
b) This integral converges because

$$\int_0^{\infty} \frac{1}{1+x^2} dx = \lim_{b\to\infty} \int_0^{b} \frac{1}{1+x^2} dx$$

$$= \lim_{b\to\infty} [\arctan x]_0^b = \lim_{b\to\infty} (\arctan b - \arctan 0) = \frac{\pi}{2}$$

Integral Test for Convergence

Torrecelli's Trumpet is an object with infinite surface area but finite volume.



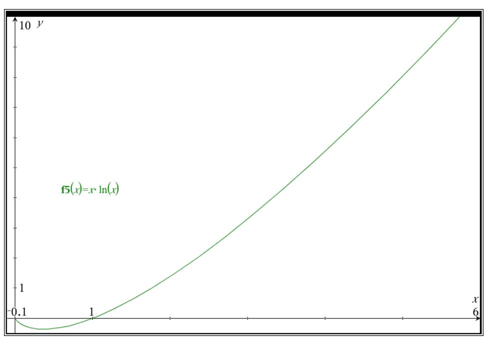
It is defined by $\int_1^{\infty} \frac{1}{x^2}dx = \lim_{b\to\infty}\int_1^{b} \frac{1}{x^2}dx = 1$. We can compare the function $\frac{1}{x^2}$ to the series $\sum_{n=1}^{\infty}\frac{1}{n^2} = 1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}+\cdots$

Integral Test: Let *f(x)* be a continuous, positive, and decreasing function. Then the series $f(1)+f(2)+f(3)+\cdots+f(n)+\cdots$ **converges if the improper integral** $\int_1^{\infty} f(x)\,dx$ **converges, and diverges if the integral diverges.**

EXAMPLE: Using the integral test to determine if a series converges

Let $u_n = \frac{1}{2\ln 2}+\frac{1}{3\ln 3}+\frac{1}{4\ln 4}+\cdots+\frac{1}{n\ln n}+\cdots$

First, examine the necessary condition for convergence, that $\lim_{n\to\infty}\frac{1}{n\ln n} = 0$. Since $f(x) = x\ln x$ is increasing for all values of $x \geq 2$, the function $f(x) = \frac{1}{x\ln x}$ is positive and decreasing.



$$\int_2^{\infty}\frac{1}{x\ln x}dx = \lim_{b\to\infty}\int_2^{b}\frac{1}{x\ln x}$$

$$u = \ln x, du = \frac{1}{x}dx$$

$$\lim_{b\to\infty}\int_{\ln 2}^{\ln b}\frac{1}{u}du = \lim_{b\to\infty}\left[\ln u\right]_{\ln 2}^{\ln b} = \lim_{b\to\infty}\left(\ln(\ln b)-\ln(\ln 2)\right) = \infty$$

Since the integral diverges, the sequence also diverges.

The p-Series Test:

The integral test can be used to determine convergence of series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$**,**

where *p* is a real constant. Such a series is called a *p*-series.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p > 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ diverges if } p < 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ diverges if } p = 1$$

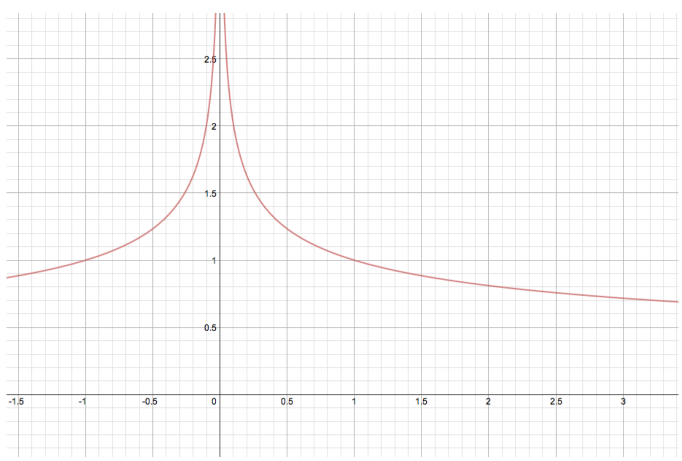
Using the integral test,

$$\text{When p} = 1,\ \lim_{b\to\infty} \int_1^b \frac{1}{x} dx = \lim_{b\to\infty} [\ln x]_1^b = \lim_{b\to\infty} (\ln b - 0) = \infty$$

$$\text{When p} > 1,\ \lim_{b\to\infty} \int_1^b \frac{1}{x^p} dx = \lim_{b\to\infty} \int_1^b x^{-p}\, dp = \lim_{b\to\infty} \left[\frac{1}{-p+1} x^{-p+1} \right]_1^b = \frac{1}{p-1}$$

Using a Comparison Test

$$\text{When } 0 \le \text{p} < 1,\ \frac{1}{n^p} \ge \frac{1}{n} \text{ which we know to diverge.}$$



Ratio test for convergence:

Let $a_n > 0$ **for** $n \geq 1$ **and** $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ **. Then**

$$\sum_{n=1}^{\infty} a_n \text{ converges is L} < 1 \text{ and diverges if L} > 1.$$

If *L* =1, the test is inconclusive.

For example, if we apply the ratio test to the harmonic series, we need to evaluate

$$\lim_{n\to\infty}\left(\frac{\frac{1}{n+1}}{\frac{1}{n}}\right) = \lim_{n\to\infty}\left(\frac{n}{n+1}\right) = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right) = 1$$

However, we know that the harmonic series diverges.

Let us now apply the ratio test to the series $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$ **.**

$$\sum_{n=1}^{\infty}\frac{1}{n^2} \quad = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty}\left(\frac{n^2}{(n+1)^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^2 = 1$$

However, we know this series converges by the p-series test.

EXAMPLE: Determine the convergence or divergence of the following series:

a) $$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty}\left(\frac{n+1}{2^{n+1}}\cdot\frac{2^n}{n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\cdot\frac{2^n}{2^{n+1}}\right) = \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \frac{1}{2}$$

Since L < 1, the series converges

b) $$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

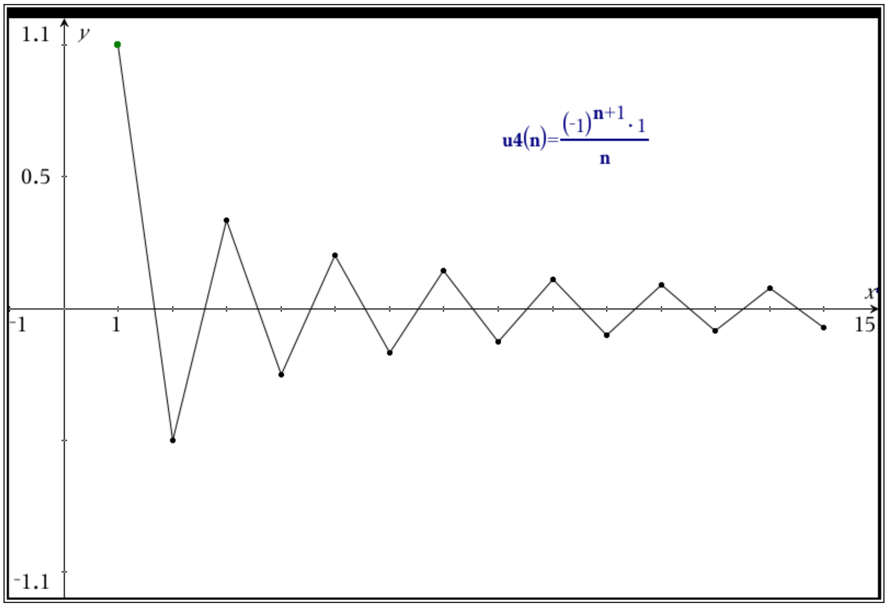
$$\sum_{n=1}^{\infty} \frac{2^n}{n!} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n\to\infty}\left(\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n+1}\cdot\frac{2^{n+1}}{2^n}\right) = \lim_{n\to\infty}\left(\frac{2}{n+1}\right) = 0$$

Since L < 1, the series converges

Alternating Series Test

$$\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{1}{k}\right)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$



If for an alternating series $\sum_{k=1}^{\infty}(u_k)$:

- $|u_{k+1}|<|u_k|$ for sufficiently large values of k
- $\lim_{k\to\infty}|u_k|=0$

then the series is convergent.

Determine whether $\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{1}{k}\right)$ is convergent.

$$|u_k|=\frac{1}{k},\ |u_{k+1}|=\frac{1}{k+1}$$

$$0<\frac{1}{k+1}<\frac{1}{k}\text{ for all positive integers }\therefore$$

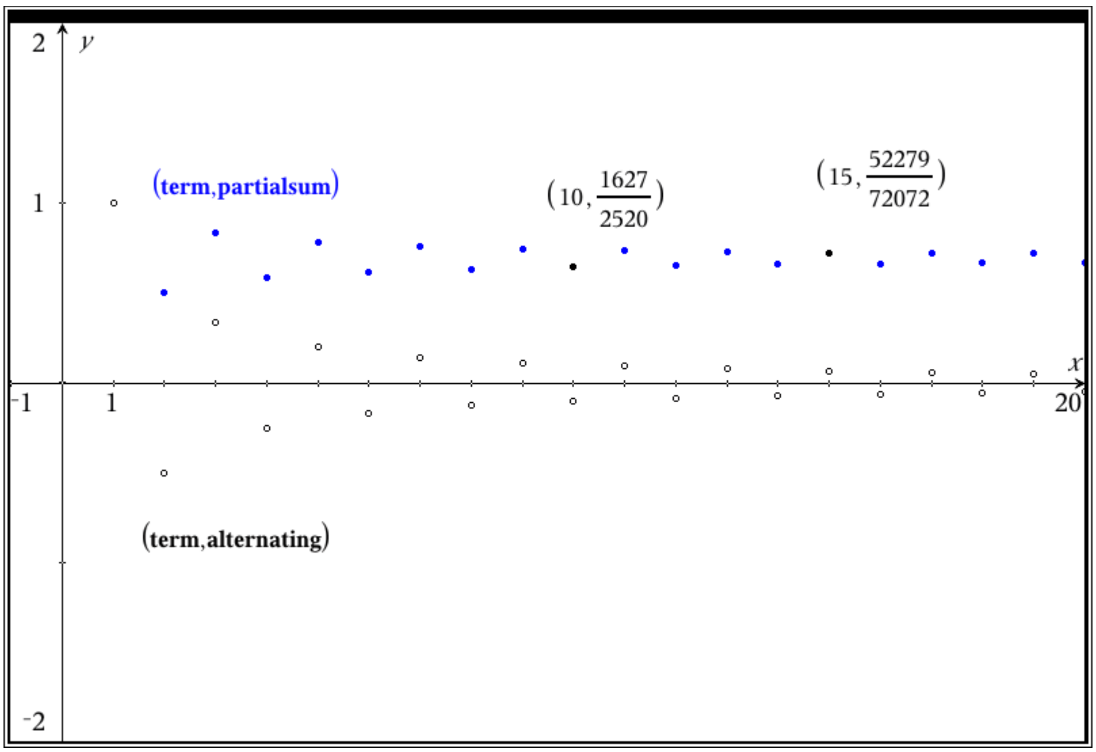
$$|u_{k+1}|<|u_k|\text{ for all positive integers}$$

$$\lim_{k\to\infty}|u_k|=\lim_{k\to\infty}\frac{1}{k}=0$$

Therefore, by the Alternating Series Test $\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{1}{k}\right)$ is convergent.

Once we know our alternating series is convergent, we can go on to think about an approximation for the limit. If we were to stop, or truncate, this partial sum at the n^{th} partial sum, how accurate would this sum be to the true limit, *S*? The distance between the n^{th} partial sum and *S*, $|S - S_n|$, known as the truncation error, is less than the distance between the $(n+1)^{th}$ partial sum and the n^{th} partial sum $|S_{n+1} - S_n|$.

$$S_{n+1} = u_1 + u_2 + u_3 + \cdots + u_n + u_{n+1}$$
$$S_n = u_1 + u_2 + u_3 + \cdots + u_n$$
$$S_{n+1} - S_n = u_{n+1}$$



If $S = \sum_{k=1}^{\infty}(u_k)$ is the sum of an alternating series that satisfies:

- $|u_{k+1}| < |u_k|$ for positive integers, *k*
- $\lim_{k\to\infty}|u_k| = 0$

then the error in taking the first *n* terms as an approximation of the sum, *S* – the truncation error – is less than the absolute value of the $(n + 1)^{th}$ term.

$$|S - S_n| < |u_{n+1}|$$

For example, taking the first 9 terms of the alternating harmonic series will give an error less than the 10^{th} term for an approximation of the limit.

$$\left|S - S_9\right| < \left|u_{10}\right| = \left|\frac{-1}{10}\right| = 0.1$$

Taking 100 terms, $\left|S - S_{99}\right| < \left|u_{100}\right| = \left|\frac{-1}{100}\right| = 0.01$