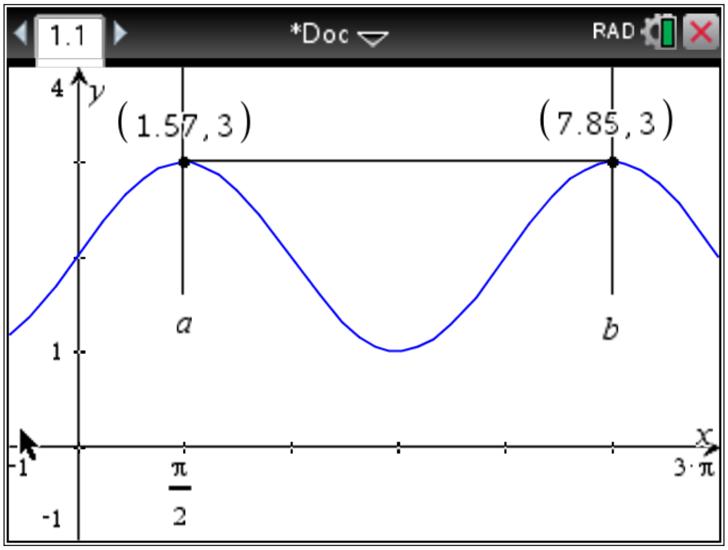
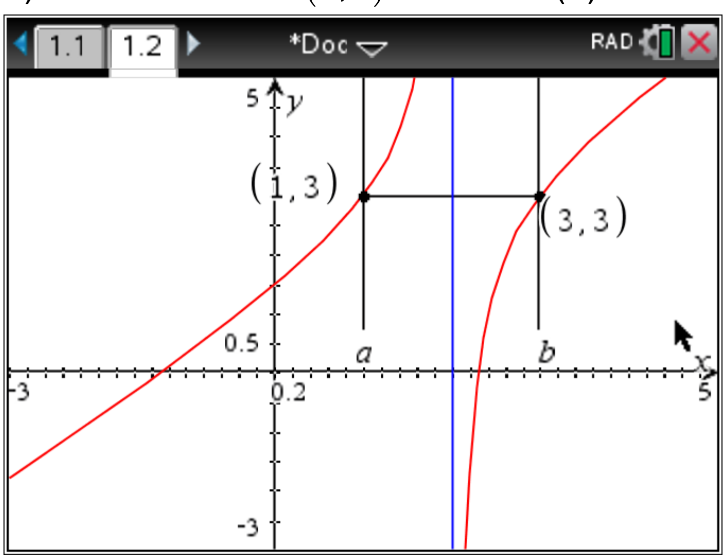
**Rolle's Theorem and the Mean Value Theorem**

For a non-constant function on an interval [*a,b*], if we know that the function is continuous and differentiable and it starts and finishes at the same *y*-value, it is clear that there must be at least one turning point somewhere in the interval.

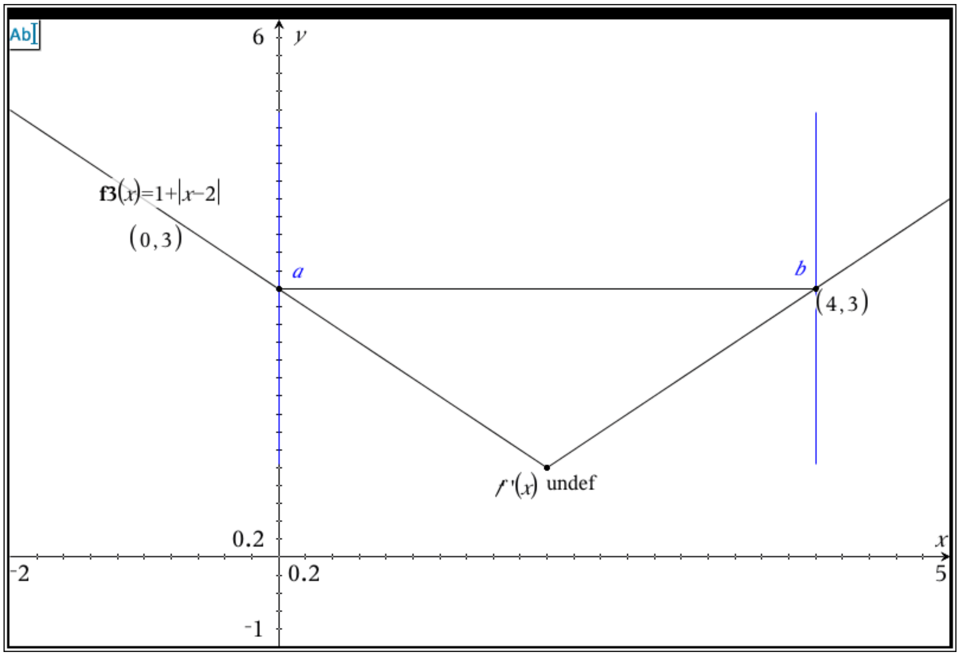



**Rolle's Theorem:** For a function, $f(x)$, that is continuous on an interval [*a,b*] and differentiable on the interval (*a,b*), if $f(a)=f(b)$ then there must exist some point, $c \in (a,b)$ such that $f'(c)=0$.

Continuity is important because we could have a function, say $f(x)=\dfrac{x^2-x-3}{x-2}$ for which $f(a)=f(b)$ but there is no $c \in (a,b)$ such that $f'(c)=0$.



We also need differentiability because we could have a continuous function that does not have a turning point, for example $f(x) = 1 + |x - 2|$.




The most common application of Rolle's Theorem is to establish the maximum number of roots of a polynomial.

EXAMPLE: Prove that the polynomial $f(x) = x^3 + 3x^2 + 6x + 1$ has exactly one root.

By the Intermediate Value Theorem, if sign of the value of the function changes from negative to positive or from positive to negative, there must be a value in that interval where $f(x_0) = 0$.

$$f(-1) = -3 < 0$$
$$f(1) = 11 > 0$$

Since all polynomials are continuous, there must exist an $x_0 \in (-1,1)$ such that $f(x_0) = 0$, therefore *f(x)* has at least one root.

We now need to show that there is only one root, so suppose there are two and look for a contradiction.
If there existed an $x_1 > x_0$, then $f(x_0) = f(x_1) = 0$ which meets the conditions of Rolle's Theorem, that the function is continuous on an interval [*a,b*] and differentiable

on the interval (*a,b*), if $f(a)=f(b)$ then there must exist some point, $c\in(a,b)$ such that $f'(c)=0$.

$$\begin{aligned} &f'(x)=3x^2+6x+6\\ &3(x^2+2x+2)=0\\ &b^2-4ac=4-8=-4<0 \end{aligned}$$

Therefore, there are no real roots of the derivative and we have a contradiction. There is only one real root, $x_0$.

EXAMPLE: Given $f(x)=\cos(2x)+2\cos x, 0\le x\le 2\pi$, use Rolle's Theorem to show that the equation $f'(x)=0$ has at least one solution on the interval $]0,2\pi[\equiv(0,2\pi)$ (The notation $]\ [$ means an open, rather than closed, interval – endpoints not included in the interval). Hence, find all the solutions to $f'(x)=0$ on $]0,2\pi[$ and verify your answer with your GDC.

First verify the conditions of Rolle's Theorem, *f(x)* is differentiable and continuous on the interval $]0,2\pi[$.

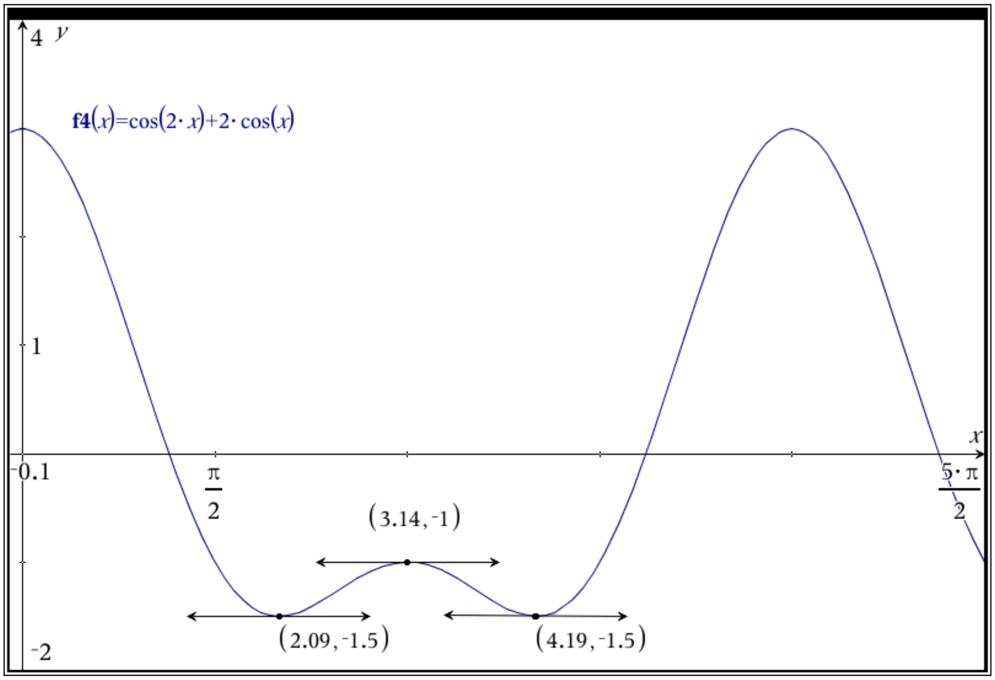
Also,

$$\begin{aligned} &f(0)=\cos(2\cdot 0)+2\cos 0=1+2=3\\ &f(2\pi)=\cos(2\cdot 2\pi)+2\cos 2\pi=1+2=3 \end{aligned}$$

so $f(0)=f(2\pi)$.

By Rolle's Theorem, there is at least one $x\in\ ]0,2\pi[$ such that $f'(x)=0$

$$\begin{aligned} &f'(x)=-2\sin(2x)-2\sin x=0\\ &-2(\sin(2x)+\sin x)=0\\ &2\sin x\cos x+\sin x=0\\ &\sin x(2\cos x+1)=0\\ &\sin x=0, x=\pi\\ &\cos x=-\frac{1}{2}, x=\frac{2\pi}{3}, x=\frac{4\pi}{3} \end{aligned}$$

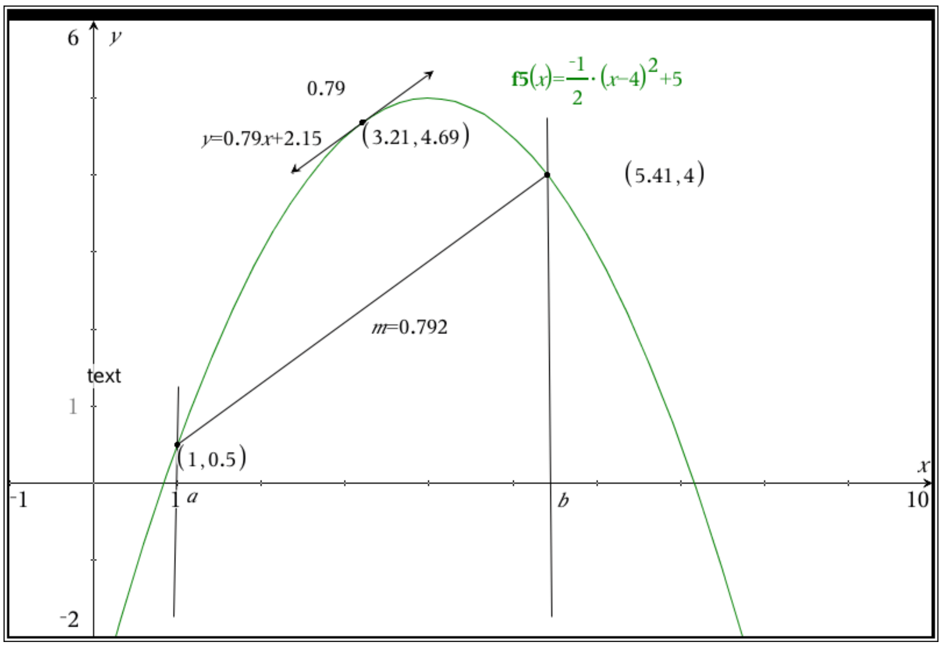


y
4
f4(x)=cos(2· x)+2· cos(x)
1
x
-0.1
π/2
5·π/2
(3.14, -1)
(2.09, -1.5)
(4.19, -1.5)
-2

Rolle's Theorem is important because it easily allows us to prove a fundamental result: The Mean Value Theorem (MVT).

**Mean Value Theorem:** Let *f* be a function that is continuous on $]a,b[$ and differentiable on $]a,b[$. Then, there is at least one $x \in ]a,b[$ such that

$$f'(x) = \frac{f(b)-f(a)}{b-a}.$$




**Proof of MVT:**

Consider the linear function $h(x) = \frac{f(b)-f(a)}{b-a}(x-a)$ and the function $g(x) = f(x)-h(x)$. Because *f(x)* was already identified as a function that is continuous and differentiable and *h(x)* is a linear function that is always continuous and differentiable, *g(x)* is also continuous and differentiable.

$$g(a) = f(a) - \frac{f(b)-f(a)}{b-a}(a-a) = f(a)$$

$$g(b) = f(b) - \frac{f(b)-f(a)}{b-a}(b-a) = f(b)-(f(b)-f(a)) = f(a)$$

$$g(b) = g(a)$$

Therefore, by Rolle's Theorem there exists some point, $c \in (a,b)$ such that $f'(c) = 0$.

$$g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$$
$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

There are a number of applications of the MVT.

EXAMPLE: Prove that $|\sin a - \sin b| \le |a-b|$.

Since *f(x)* = sin(*x*) is continuous and differentiable for all values of *x*, by the Mean Value Theorem there exists $c \in (a,b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.

$$\frac{\sin(b)-\sin(a)}{b-a} = \cos(c)$$
$$\left|\frac{\sin(b)-\sin(a)}{b-a}\right| = |\cos(c)| \le 1$$
$$\Rightarrow |\sin a - \sin b| \le |a-b|$$

Note that $|a-b| = |b-a|$

EXAMPLE: Show that $\sqrt{1+h} < 1 + \frac{h}{2}$ for any $h > 0$.

Given $h > 0$, let $f(x) = \sqrt{1+x}, 0 \le x \le h$. $f(x)$ is continuous on $]0,h[$ and $f(x)$ is differentiable on $]0,h[$.

$$f'(x) = \frac{1}{2\sqrt{1+x}}, 0 < x < h$$

By the MVT, there is at least one value of $x \in ]0,h[$ such that

$$f'(x) = \frac{f(h)-f(0)}{h-0} \Rightarrow \frac{1}{2\sqrt{1+x}} = \frac{\sqrt{1+h}-1}{h}$$

$$f'(x) > 0$$
$$x \in I \subseteq D_f$$
$$f(x) = g(x) + c$$
$$[a,b]$$
$$0 = f'(x) = \frac{f(b) - f(a)}{b - a} \Rightarrow f(b) = f(a)$$
$$0 < x < h \Rightarrow 1 + x > 1 \Rightarrow \sqrt{1+x} > 1$$
$$\frac{1}{\sqrt{1+x}} < 1 \Rightarrow \frac{1}{2\sqrt{1+x}} < \frac{1}{2}$$
$$\therefore \frac{\sqrt{1+h} - 1}{h} < \frac{1}{2} \Rightarrow \sqrt{1+h} - 1 < \frac{1}{2}h \Rightarrow \sqrt{1+h} < 1 + \frac{1}{2}h$$

EXAMPLE: If *f(x)* is such that $f(2) = -4$ and $f(x) \geq -2$ for all $x \in ]2,7[$, find the smallest possible value for $f(7)$.

Since the derivative exists for all $x \in ]2,7[$, we know that *f(x)* is differentiable and hence continuous. By the Mean Value Theorem, there exists some $c \in ]2,7[$ such that $f'(c) = \frac{f(7) - f(2)}{7 - 2}$. Since $f(x) \geq -2$ for all $x \in ]2,7[$

$$\frac{f(7) - f(2)}{5} \geq -2$$
$$f(7) + 4 \geq -10$$
$$f(7) \geq -14$$

so, the smallest possible value for $f(7) = -14$.

EXAMPLE: A car driving along the highway and travelling below the speed limit of 70 mph passes a police officer at 12:00. At 12:20, the car passes another police officer 24 miles down the road; again, it was travelling at less than 70 mph. The driver is pulled over by the policeman, a math major in college, and given a ticket.

Use the MVT to show how the police officer knew that the driver had exceeded the speed limit during his journey.

Let *f(t)* be the function that gives the car's position at time *t* hours after 12:00.

Assuming this function to be continuous and differentiable, by the MVT, there is a time, $t_0$ at which

$$f'(t_0) = \frac{f\left(\frac{1}{3}\right) - f(0)}{\frac{1}{3} - 0}$$

$$\frac{24 - 0}{\frac{1}{3}} = 72$$

However, $f'(t_0)$ is the car's velocity at some time $t_0$ so at some time during his journey, the car must have been travelling at 72 mph.

**Corollaries to the Mean Value Theorem**

**Corollary 1:** If $f'(x) = 0$ for all $x \in I \subseteq D_f$ (all *x* are members of the set *I* which is a proper subset of all of the elements in the domain of *f*), then *f* is constant on the interval ***I.***

Proof: Consider any two points $a < b$ in ***I*** on the interval $[a,b]$. Then,

$$0 = f'(x) = \frac{f(b) - f(a)}{b - a} \Rightarrow f(b) = f(a).$$

Thus, the function is constant on ***I***.

**Corollary 2:** If $f'(x) = g'(x)$ for all $x \in I \subseteq D_f \cap D_g$, then $f(x) = g(x) + c$

**Corollary 3:** If $f'(x) > 0$ for all $x \in I \subseteq D_f$, then *f* is increasing on the interval ***I.***

Proof: Consider any two points $a < b$ in ***I*** on the interval $[a,b]$.

$$f'(x) = \frac{f(b) - f(a)}{b - a} > 0 \Rightarrow f(b) - f(a) > 0 \Rightarrow f(b) > f(a)$$

Therefore for any two values $b > a \Rightarrow f(b) > f(a)$, meaning that *f* is increasing on the interval ***I.***

**Corollary 4:** If $f'(x) < 0$ for all $x \in I \subseteq D_f$, then *f* is decreasing on the interval ***I.***