Separable Variables, differential equations, and graphs of their solutions

This will be an exploration of a variety of problems that occur when studying rates of change. Many of these problems can be modeled by first order differential equations with separable variables. These equations can be written in the form $\frac{dy}{dx} = \frac{f(x)}{g(y)}$.

These need to be rearranged as $g(y)\frac{dy}{dx} = f(x)$, recalling the Chain Rule and the fact that $y = y(x)$. Then, $g(y(x))\frac{dy}{dx}dx = f(x)dx$ and $\int g(y)dy = \int f(x)dx + c$.

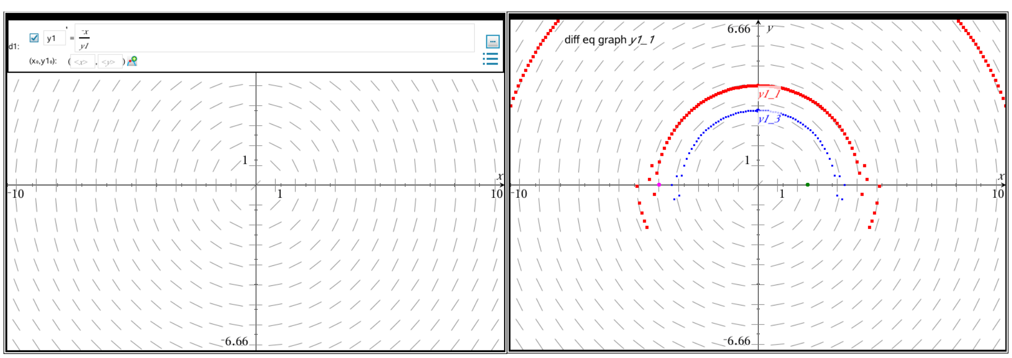
EXAMPLE: Solve the following separable differential equation $\frac{dy}{dx} = -\frac{x}{y}$

$$\frac{dy}{dx} = -\frac{x}{y} \Rightarrow ydy = -xdx$$

$$\int y\,dy = -\int x\,dx$$

$$= \frac{y^2}{2} + \frac{x^2}{2} = c$$

A Slope-Field demonstrates the gradient to the curve described by $y = y(x)$.



EXAMPLE: Show that the general solution to the differential equation $\frac{dy}{dx} = xy - x$ can be written as $y = 1 + Ae^{x^2}$ if $y > 1$.

$$\frac{1}{y-1}dy = xdx$$

$$\int \frac{1}{y-1}dy = \int xdx$$

$$\ln|y-1| = \frac{x^2}{2} + c$$

$$|y-1| = e^{\frac{x^2}{2}+c}$$

$$\text{since } y-1>0, y-1 = e^{\frac{x^2}{2}+c} = e^{\frac{x^2}{2}}e^{c}$$

$$\therefore y = 1 + Ae^{x^2}$$

EXAMPLE: Solve the differential equation $\cos^2 x\frac{dy}{dx} = e^{-2y}$ for $0 \le x < \frac{\pi}{2}$ given that $x = \frac{\pi}{4}, y = \frac{1}{2}$ giving your answer in the form $y = f(x)$.

$$\frac{1}{e^{-2y}}dy = \sec^2 xdx$$

$$e^{2y}dy = \sec^2 xdx$$

$$\int e^{2y}dy = \int \sec^2 xdx$$

$$\frac{1}{2}e^{2y} = \tan x + c \Rightarrow e^{2y} = 2\tan x + c$$

$$e = 2 + c$$

$$c = e - 2$$

$$e^{2y} = \tan x + e - 2$$

$$2y = \ln|\tan x + e - 2|$$

$$y = \frac{1}{2}\ln|\tan x + e - 2|$$

Modelling Growth and Decay phenomena

An important class of separable differential equation has the form $\frac{dy}{dt} = \pm ky$ where $k > 0$ and t is the independent variable, *time.*

$$\frac{dy}{dt} = \pm ky \Rightarrow \frac{dy}{y} = \pm kdt$$
$$\ln|y| = \pm kt + c$$
$$|y| = e^{\pm kt + c} \Rightarrow y = Ae^{\pm kt}$$

The general solution $y = Ae^{\pm kt}$ is called the exponential growth curve. This model is used to calculate continuous compound interest as well as to represent the exponential decay curve, such as in radio carbon dating, used to measure age by determining the amount of C^{14} remaining. The half-life of carbon is 5568 years.

EXAMPLE: The Lascaux Caves in southwest France contain some of the oldest and finest prehistoric art in the world. By means of complex chemical analysis on charcoal taken from the caves, scientists were able to determine that the charcoal contained 15% of the original amount of C^{14} that it would have contained when the tree it was made from was cut down.

a) Find the value of the decay constant k for the model that gives the quantity Q of C^{14} present in the sample of charcoal.

b) Hence, find an approximate age for the Lascaux paintings.

a)

$$Q(t) = Q_0 e^{-kt}$$
$$Q(5568) = Q_0 e^{-5568k} = \frac{1}{2} Q_0$$
$$5568k = \ln 2 \Rightarrow k = \frac{\ln 2}{5568} \approx 0.000124$$

b)

$$Q_0 e^{-kt} = 0.15 Q_0 \Rightarrow t = -\frac{\ln(0.15)}{k}$$
$$\Rightarrow t \approx 15200 \text{ years}$$

TRY THIS

Artjom has been exploring economic models for inflation. He decides to consider a model where the inflation decreases over time according to the model $\frac{dp}{dt}=\frac{I_0}{I+t}P$ where *t* is the time in years, *P(t)* is the price of an item at a time *t* and I_0 is the initial inflation rate.

a) Assuming that the inflation rate starts at 3%, calculate the price of an item in 10 years if *P(0)=100 €*.

b) Find an expression for *P(10)* in terms of I_0 given that *P(0)=100 €*.

John thinks that actually the inflation will increase in the near future and suggests a different model: $\frac{dp}{dt}=(0.01+0.001t)P$

c) Solve the differential equation and find the value of *P(10)* given that *P(0)=100 €*.

FIRST ORDER EXACT EQUATIONS AND INTEGRATING FACTORS

The equation $x^2\frac{dy}{dx}+2xy=1$ is an example of an **exact equation**. It is a linear differential equation as it can be written as $\frac{dy}{dx}+\frac{2y}{x}=\frac{1}{x^2}$. You may also notice that $x^2\frac{dy}{dx}+2xy=\frac{d}{dx}\left(x^2y\right)$ which means that the equation can be written as $\frac{d}{dx}\left(x^2y\right)=1$.
Integrating with respect to *x*: $\left(x^2y\right)=x+c\Rightarrow y=\frac{c}{x^2}+\frac{1}{x},X\neq 0$.

Equations that can be written in the form $\frac{d}{dx}\left(u(x)\cdot y\right)=v(x)$ are called *first order exact equation*. Most first order equations are not exact, but can be transformed by multiplying both sides of the equation by an appropriate expression called an **integrating factor.**

For example, $xy\frac{dy}{dx}+y^2=3x$ is not exact but if you multiply both sides by 2*x* you get

$$2x^2y\frac{dy}{dx}+2xy^2=6x^2$$

$$\frac{d}{dx}\left(x^2y^2\right)=6x^2$$

$$\frac{dy}{dx}+p(x)y=q(x)$$

$$\frac{d}{dx}e^{\int p(x)dx}y=e^{\int p(x)dx}q(x)$$

or $\frac{d}{dx}\left(x^2y^2\right)=6x^2$ which is exact. If we can find an integrating factor that makes a first order equation exact, then the problem of solving it becomes simply an integration problem.

Fortunately, for first order linear equations of the form $\frac{dy}{dx}+p(x)y=q(x)$, we can find the integrating factor in a systematic way as stated in the following theorem:

Theorem: Given the differential equation $\frac{dy}{dx}+p(x)y=q(x)$, the function $I(x)=e^{\int p(x)dx}$ is an integrating factor that transforms this differential equation into and exact differential equation of the form $\frac{d}{dx}e^{\int p(x)dx}y=e^{\int p(x)dx}q(x)$

Proof: Multiply both sides by *I(x):*

$$e^{\int p(x)dx}\frac{dy}{dx}+p(x)e^{\int p(x)dx}y=e^{\int p(x)dx}q(x)$$

$$\frac{d}{dx}e^{\int p(x)dx}=\left(\frac{d}{dx}\int p(x)dx\right)\cdot e^{\int p(x)dx}=p(x)e^{\int p(x)dx}$$

$$e^{\int p(x)dx}\frac{dy}{dx}+p(x)\cdot e^{\int p(x)dx}y=\frac{d}{dx}e^{\int p(x)dx}\cdot y$$

$$\frac{d}{dx}e^{\int p(x)dx}y=e^{\int p(x)dx}q(x)$$

$$\frac{d}{dx}e^{\int p(x)dx}=\left(\frac{d}{dx}\int p(x)dx\right)\cdot e^{\int p(x)dx}=p(x)e^{\int p(x)dx}$$

$$e^{\int p(x)dx}\frac{dy}{dx}+p(x)\cdot e^{\int p(x)dx}y=\frac{d}{dx}e^{\int p(x)dx}\cdot y$$

$$\frac{d}{dx}e^{\int p(x)dx}y=e^{\int p(x)dx}q(x)$$

EXAMPLE: Consider the first order linear equation $x\frac{dy}{dx}+3y=e^{x^3}$

a) Find an integrating factor for this differential equation.
b) Hence, solve the differential equation.

a)

$$x\frac{dy}{dx}+3y=e^{x^3}\Rightarrow\frac{dy}{dx}+\frac{3y}{x}=\frac{e^{x^3}}{x},x\neq 0$$

$$I(x)=e^{\int\frac{3}{x}dx}=e^{3\ln x}=x^3$$

b)

$$x^3\frac{dy}{dx}+x^3\cdot\frac{3y}{x}=x^3\cdot\frac{e^{x^3}}{x}$$

$$\frac{d}{dx}\left(x^3y\right)=x^2\cdot e^{x^3}$$

$$\Rightarrow x^3y=\frac{1}{3}e^{x^3}+c$$

$$y=\frac{1}{3x^3}e^{x^3}+\frac{c}{x^3},x\neq 0$$

EXAMPLE: Find the particular solution of the following first order differential equation that satisfies the initial condition, given

$\frac{dy}{dx}+2y=e^x$ and $y(0)=1$

$$e^{2x}\left(\frac{dy}{dx}+2y\right)=e^{2x}e^x$$

$$e^{2x}\frac{dy}{dx}+2e^{2x}y=e^{3x}$$

$$\frac{d}{dx}e^{2x}y=e^{3x}\Rightarrow e^{2x}y=\frac{1}{3}e^{3x}+c$$

$$y=\frac{1}{3}e^x+\frac{c}{e^{2x}}=\frac{1}{3}e^x+ce^{-2x}$$