SEQUENCES

A **sequence** is a mathematical function whose domain is the set of integers. In the Calculus option, we are concerned with the limit of a sequence to determine whether or not it converges or diverges

The **limit** of a sequence is defined as follows: Let L be a real number. The limit, $\lim_{n\to\infty} a_n = L$, if for each $\mathcal{E} > 0$, there exists *M > 0* such that $|a_n - L| < \mathcal{E}$, whenever *n > M*. Sequences that have limits are said to **converge**, whereas sequences that do not have limits **diverge**.

THEOREM: Let *f* be a function of a real variable such that $\lim_{x\to\infty} f(x) = L$. If [$a_n$] is a sequence such that *f(n)* = a_n for every positive integer *n*, , then $\lim_{n\to\infty} a_n = L$.

EXAMPLE: By the theorem above, if $a_n = \left(1+\frac{1}{n}\right)^n$, $\lim_{n\to\infty} a_n = e$, because $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$.

EXAMPLE: Show that the sequence whose *n*th term is $a_n = \frac{n^2}{2^n-1}$ converges.

$f(x) = \frac{x^2}{2^x-1}$ is indeterminate. Using L'Hôpital's rule, $f'(x) = \frac{2x}{2^x(ln 2)}$ is also indeterminate. $\lim_{x\to\infty}\frac{x^2}{2^x-1} = \lim_{x\to\infty}\frac{2x}{2^x(\ln 2)} = \lim_{x\to\infty}\frac{2}{2^x(\ln 2)^2} = 0$. Therefore, by the theorem above, $a_n = \frac{n^2}{2^n-1}$ converges.

SQUEEZE THEOREM FOR SEQUENCES

If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$ and there exists and integer *N* such that $a_n \le c_n \le b_n$, for all *n > N*, then $\lim_{n\to\infty} c_n = L$.

For example, show that $c_n = (-1)^n\frac{1}{n!}$ converges and find its limit.

$$n! = 1\cdot2\cdot3\cdot4\cdot5\cdot6\cdots n = 24\cdot5\cdot6\cdot\cdots\cdot n \text{ and,}$$
$$2^n = 2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot\cdots\cdot2 = 16\cdot2\cdot2\cdot\cdots\cdot2$$

For $n \ge 4, 2^n < n!$ so, $\frac{-1}{2^n} \le (-1)^n\frac{1}{n!} \le \frac{1}{2^n}$. Therefore, by the Squeeze Theorem,

$$\lim_{n\to\infty}(-1)^n\frac{1}{n!} = 0.$$

ABSOLUTE VALUE THEOREM

For the sequence [a_n], if $\lim_{n\to\infty}|a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

MONOTONIC SEQUENCE: A sequence is monotonic if its terms are nondecreasing, $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$ or if its terms are nonincreasing, $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$.

BOUNDED SEQUENCE:

1. A sequence is **bounded above** if there is a real number, *M* such that $a_n \leq M$ for all *n*. The number *M* is called the **upper bound** of the sequence.
2. A sequence is **bounded below** if there is a real number, *N* such that $a_n \leq N$ for all *n*. The number *N* is called the **lower bound** of the sequence.
3. A sequence is **bounded** if it is bounded above and below.

BOUNDED MONOTONIC SEQUENCE: If a sequence is bounded and monotonic, then it converges.

EXAMPLES:

1. The sequence $a_n = \frac{1}{n}$ is bounded and monotonic, therefore it converges.
2. The divergent sequence $b_n = \frac{n^2}{(n+1)}$ is monotonic but not bounded. It is bounded below.
3. The divergent sequence $a_n = (-1)^n$ is bounded but not monotonic.